Fractional Quantum Hall Effect and vortex lattices.
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introduction

The experimental discovery of Integer Quantum Hall Effect (IQHE) by v.Klitzing (1980)and Fractional Quantum Hall Effect (FQHE) by Tsui,Stormer and Gossard (1982) was one of the outstanding achievements in condensed matter physics of the last century. The qualitative physical explanation of IQHE was given soon after experimental discovery but the theory of FQHE is far from being complete. In pioneer works of Laughlin (1981,1983) the variational function for some odd LI fillings was suggested. He put forward the idea of the excitations with the fractional electron charge.

other fractions

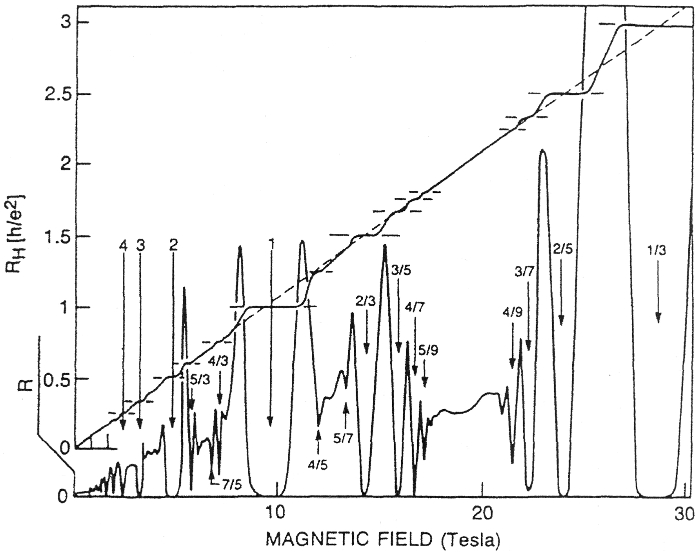
The other fractions existence was explained by the construction of the daughter states (Haldane 1983, Laughlin 1984, B.Halperin 1984). The most successful phenomenological description was given by Jains (1989-1990) model of "composite" fermions giving the majority of the observed fractions. According to Jains model 2d electrons in the perpendicular magnetic field are "dressed" by 2 flux quanta of magnetic field opposite to the external one. The inclusion of this additional field in a formalized theory gives Chern-Simons Hamiltonian (B.Halpern, P.Lee, N.Read, 1993)

However this model introduces artificial 6-fermionic model and there are experimental fractions which can not be found by this model. QHE consists in the existence of the plateau of Hall conductivity near some special values of the electron density. At integer fillings of LI we have large energy gaps in high magnetic field with the macroscopic degeneracy of the levels. That results in the electron motion along lines where the impurity potential is constant. Those lines are closed near the minima or maxima of impurity potential and therefore the corresponding electron states have not macroscopic current.

plateau 2

If the electron chemical potential is somewhere between LI the density is changed due to localized states without changing of the Hall current defined by the delocalized states near the percolation threshold. We see that for IQHE it is important the existence of large energy gaps and localized states developed in the weak impurity potential due to the degeneracy of the one particle states. The experimental data near the fractional fillings are quite close to those at integer fillings. Therefore the general physical picture must be close for IQHE and FQHE.

picture



R_H [h/e^2]
R
3
2.5
2
1.5
1
0.5
0
0
4
3
2
1
5/3
4/3
7/5
4/5
5/7
2/3
3/5
4/7
5/9
4/9
3/7
2/5
1/3
10
20
30
MAGNETIC FIELD (Tesla)

extension

I shall show how to remove the restrictions of the Jains model and to obtain a more general and more simple model with the standard Coulomb interaction. The main conception is associated with the possibility to have topological textures in 2d electron system. The vortices are wide spread in condensed matter physics. The simple and general definition can be done using the canonical transformation for the field operators of second quantization
$\psi(\mathbf{r}) = e^{i\alpha(\mathbf{r})}\chi(\mathbf{r})$ $\psi^{+} = \chi^{+}e^{-i\alpha}$
with $\nabla\alpha$ having vortex like singularities.

$curl \nabla\alpha = 2\pi K \sum_{n,n'} \delta(\mathbf{r} - \mathbf{r_{nn'}})$ where $\mathbf{r}_{nn'} = n\vec{\tau}_1 + n'\vec{\tau}_2$ form a lattice, K is some integer. I assume full spin polarization and omit spin indices. It is evident that both ψ,ψ^+ and χ,χ^+ satisfy Fermi commutation relations. The problem to find $\nabla\alpha$ and $e^{i\alpha}$ can be solved by Weierstrass ζ function.

zeta2

It is a convergent series

$$\zeta = \frac{1}{z} + \sum_{T_{nn'} \neq 0} \left(\frac{1}{z - T_{nn'}} + \frac{1}{T_{nn'}} + \frac{z}{T^2_{nn'}} \right)$$

where $z = x + iy$, $T_{nn'} = n\tau_1 + n'\tau_2$ and τ_1, τ_2 are the minimal complex periods of the vortex lattice.

In the real representation $\nabla\alpha = K(Re\zeta, Im\zeta)$ and
$\alpha = K\int_{\mathbf{r_0}}^{\mathbf{r}}(Re\zeta dx + Im\zeta dy)$
The phase factor $e^{i\alpha}$ will be simple function on 2d plain for any integer K. The substitution of the above expressions into the standard hamiltonian of the interacting electrons in magnetic field gives the new hamiltonian

hamiltonian

$H = \frac{\hbar^2}{2m} \int \chi^+ \left(-i\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}\right)^2 \chi d^2 r +$
$\frac{1}{2} \int U(\mathbf{r} - \mathbf{r}')\chi^+(\mathbf{r})|chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r}) d^2 r d^2 r'$
where $U(\mathbf{r})$ is Coulomb interaction, $\mathbf{A}(\mathbf{r})$ is the vector potential of the external uniform magnetic field. I take the gauge linear in coordinates. therefore $\mathbf{A}(\mathbf{r} + \vec{\tau}) = \mathbf{A}(\mathbf{r}) + \mathbf{A}(\vec{\tau})$
ζ-function has the same property $\zeta(z + \tau) = \zeta(z) + \delta(\tau)$.

effective potential

If we introduce "effective " vector potential $\mathbf{A}_{eff} = \mathbf{A} - \frac{c\hbar}{e}\nabla\alpha$ any translation along the periods of the vortex lattice makes a change in the effective vector-potential which can be removed by the change of the gauge. Therefore the obtained hamiltonian is invariant under magnetic translations
$T_m(\vec{\tau})\chi(\mathbf{r}) = \chi(\mathbf{r} + \vec{\tau})\exp\left(\frac{ie}{c\hbar}\mathbf{A}_{eff}(\vec{\tau})\mathbf{r}\right)$ for any real period of the vortex lattice.

flux

It is easy to connect the effective vector-potential with the "effective" magnetic flux through the unit cell of the vortex lattice

$\Phi = \oint \mathbf{A}_{eff}\, d\mathbf{R} = \mathbf{A}_{eff}(\vec{\tau}_1)\vec{\tau}_2 - \mathbf{A}_{eff}(\vec{\tau}_2)\vec{\tau}_1 = \mathbf{B}_0 \vec{\tau}_1 \times \vec{\tau}_2 + K\Phi_0$

where the flux quantum $\Phi_0 = \frac{2\pi|e|}{c\hbar}$, $\mathbf{B}_0$ is the external magnetic field normal to 2d plain.

representation

Magnetic translations are non commutative and give ray representations of ordinary translation group. The simple finite representation and simple spectral properties are possible only for the rational number of the flux quanta per the unit sell (E.Brown, 1964),(J.Zak, 1964)

$\Phi = \frac{l}{N}\Phi_0 = B_0 s + K\Phi_0$

l, N are integers, s is the area of the unit cell.Thus the situation is isomorphic to the case of the magnetic field with the rational flux number per the unit cell plus the periodic magnetic field with the zero flux.

strong field

Consider the case of the weak interaction when kinetic one particle hamiltonian is the main term. That corresponds to the large magnetic field when Coulomb interaction grows as $\sqrt{B_0}$ and the kinetic part grows as B_0 and is possible to use a perturbation theory. Any magnetic translation gives the state with the same energy and these translations does not commute. As a consequence one has highly degenerate spectrum. In order to find the one particle states one must put some boundary conditions. The simplest are generated by magnetic translations

$T_m(\mathbf{L})\chi(\mathbf{r}) = \chi(\mathbf{r})$

here $\mathbf{L} = \mathbf{L_1}, \mathbf{L_2}$which define the sample size,like Born-v.Karman conditions.

groundstate

The simplest realization of the finite representation is the sample of $N \times N$ unit cells and the general case $M_1 N \times M_2 N$ unit cells. It must be $M_1 M_2 N^2$ states of the same energy as the consequence of the magnetic translations non commutativity. The ground state energy according to the perturbation theory is
$E_0 = N^2 M_1 M_2 \epsilon_0 + \frac{1}{2} \int U(\mathbf{r} - \mathbf{r}') < \chi^+(\mathbf{r}) \chi^+(\mathbf{r}') \chi(\mathbf{r}') \chi(\mathbf{r}) d^2 r d^2 r'$
where ϵ_0 is the ground state energy for the kinetic part and the angle brackets denote the average over the filled states. If one adds more electrons they go to the state with the larger energy.The gaps must be found numerically.

electron densities

The set of the observed electron densities depend on the value of the gaps, temperatures, and sample purity. To take into account these numerous factors is a difficult task. But it is easy to solve the opposite problem- to find the vortex lattices corresponding to the observed densities using the expression for the electron density $n_e = \frac{B_0}{\Phi_0}\frac{N}{l - NK}$.

The observed fractions are given by the following tables.

$K = -2, \quad l = 1$

q	1	2	3	-5	-2	-3	-4	4	∞
ν	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{4}{9}$	$\frac{1}{2}$

That fractions correspond to the celebrated Jain's rule. Especially must be noted the half filling of the Ll corresponding to $limN \to \infty$ and the effective flux equal to zero. In this case we have an ordinary group of the translations with a normal band structure. This state can be achieved by the increasing of electron density $N > 0$ as well as by decreasing $N < 0$. This circumstance must create the symmetry vanishing of the gap for electron and holes.

Other observed fractions correspond to the table

$K = -1, \;\; l = 1$

q	-4	4	2
ν	$\frac{4}{3}$	$\frac{4}{5}$	$\frac{2}{3}$

where one have the double of the fraction $2/3$, and also

$K = -1, \;\; l = 2$

q	-7	-5	5	2
ν	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{1}{2}$

here one have not observed double of the fraction $1/2$ with the gap. The exclusion of the doubles requires extensive numerical calculations.

end2

All observed fractions correspond to vortex lattices with 1 or 2 flux quanta per unit cell. The one particle ground state degeneracy for the periodic magnetic field was found initially in the work of B.Dubrovin and P.Novikov(1980).