Interaction-Localization Paradigm: Bose Gas in a random Environment

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Abstract. The localization-disorder paradigm is analyzed for a specific system of weakly repulsive Bose gas at zero temperature placed into a quenched random potential. We show that, at low average density or weak enough interaction the particles fill deep potential wells of the random potential whose radius and depth depend on the characteristics of the random potential and the interacting gas. The localized state is the random singlet with no long-range phase correlation. At a critical density the quantum phase transition to the coherent superfluid state proceeds. We calculate the critical density in terms of the geometrical characteristics of the noise and the gas. In a finite system the ground state becomes non-ergodic at very low density. For atoms in traps four different regimes are found, only one of it is superfluid. The theory is extended to lower (1 and 2) dimensions. Its quantitative predictions can be checked in experiments with ultracold atomic gases and other Bose-systems.

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INTRODUCTION

This article is a tribute to the memory of the great physicist Lev Davidovich Landau to centenary of his birthday. He remains unsurpassed in introduction of new, extremely general and simple notions, such as spontaneous symmetry breaking and density matrix which enlightened many different branches of physics, chemistry and biology. The famous course of theoretical physics by Landau and Lifshitz is a necessary office attribute of any physicist, experimenters as well as theorist, postdocs and graduate students as well as professors. The course was started in 1940-th and it is still demanded. The reason of this unprecedented longevity is the Landau's unique and universal view of the entire physics unifying seemingly disconnected its branches into a single science. We all are his pupils.

The interplay between interaction and disorder is an important paradigm of condensed matter physics. In 1958 Anderson[1] showed that in disordered solids a non-interacting electron may become localized due to the quantum interference. A phenomenological theory of localization[2, 3] concluded that non-interacting electrons in one and two dimensions are always localized. In three dimensions the localized and extended states are separated by the mobility edge. States with energy significantly below this edge in 3 dimensions are strongly localized. They appear in rare fluctuations of the quenched random potential[4, 5, 6]. These instanton-type states broaden and eventually overlap with growing energy. A system of non-interacting fermions in the random potential transits from the insulator to metal state when its Fermi energy exceeds the mobility edge. Thus, the Pauli principle delocalizes fermions in 3 dimensions, but leave them localized in lower dimensions. The common belief is that the repulsive interaction suppresses the localization. So far this problem was studied only in the limit of a weak disorder[7, 8]. Therefore, the interaction induced delocalization transition remains beyond the frameworks of the theory. The metal-insulator transition in 2 dimensions was observed in experiments[9] suggesting the decisive role of interaction.

The physical picture changes drastically for bosons. The non-interacting bosons condense at a single-particle state with the lowest energy. In a homogeneous system it leads to a coherent quantum state known as the Bose-Einstein condensate (BEC). Examples are superfluid phases of He[10], superconductors [11], BEC of ultra-cold alkali a toms[12, 13], excitons in semiconductors[14] and spin waves in magnets [15]. BEC still persists when a small amount of disorder is added to the system. BEC in a random environment was observed in the superfluid phase of ^{4}He in Vycor glass or aerogels[17] , in ^{3}He in aerogels [18] and in ultra-cold alkali atoms in disordered traps [19, 21, 20, 22, 23, 24, 25].

In a random environment and in the absence of interaction, all Bose-particles fall into the lowest localized

single-particle state. Such a ground state is non-ergodic since its energy and spatial extension depend on a specific realization of the disorder. An arbitrary small repulsive interaction redistributes the bosons over multiple potential wells and restores ergodicity. Hence, contrary to the fermionic case, the perturbation theory with respect to the interaction strength is invalid. At low temperature, the Bose systems display superfluidity provided the density n of bosons exceeds a critical value n_c. At either weak disorder or strong interaction, i.e. at $n \gg n_c$, the disorder corrections to the superfluid density n_s (and the condensate density n_0) are small[26, 27, 28] . These correction blow up with the interaction decreasing, signaling the breakdown of the theory.

We present an alternative approach to the problem of the interaction-induced delocalization starting from deeply localized state of the Bose-gas in a random potential. We present a simple and visual picture of the deeply localized state, which decays into remote weakly coupled fragments. We give a geometrical description of fragments and their distribution in space. At a critical density n_c, which we express in terms of the disorder characteristic and interaction strength, the increasing tunneling of particles between fragments leads to transition from the random singlet state to the coherent superfluid.

SINGLE-PARTICLE LEVELS IN AN UNCORRELATED RANDOM POTENTIAL

The random environment produces a random potential $U(\mathbf{x})$ for the bosons. In this section we assume that $U(\mathbf{x})$ is Gaussian distributed with zero average and short range correlations

$$\left\langle U(\mathbf{x}) U\left(\mathbf{x}'\right)\right\rangle = \kappa^2 \delta(\mathbf{x}-\mathbf{x}') \tag{1}$$

The probability to find the random potential $U(\mathbf{x})$ in a volume Ω is the product of independent probabilities in each point (we write it for a 3d system):

$$W_0[U(\mathbf{x})] = \int \exp\left(-\frac{1}{2\kappa^2}\int_\Omega U^2(\mathbf{x})\,d\mathbf{x}\right) \prod_{\mathbf{x}\in\Omega}\left[\left(\frac{\Delta\Omega}{2\pi\kappa^2}\right)^{1/2} dU(\mathbf{x})\right] \tag{2}$$

In the absence of interaction the single-particle wave function $\psi(\mathbf{x})$ obeys the Schrödinger equation

$$\frac{\hbar^2}{2m}\nabla^2\psi + (E - U(\mathbf{x}))\psi = 0. \tag{3}$$

Its energy levels $E[U(\mathbf{x})]$ are functionals of the potential $U(\mathbf{x})$. The only characteristic of the random potential κ together with the Planck's constant $\hbar$ and the mass m establishes the scales of length and energy:

$$\mathscr{L} = \frac{\hbar^4}{m^2\kappa^2}, \quad \mathscr{E} = \frac{\hbar^2}{m\mathscr{L}^2}, \tag{4}$$

which we call the Larkin length and Larkin energy, respectively in analogy with the scale found in the Larkin's work [29] for an elastic medium in a random field. The density of states $\nu(E)$ belonging to (3) in the limit $E < 0, |E| \gg \mathscr{E}$ was calculated in [4, 5, 6] (for a complete summary see [30]). We reproduce some of their results and extend them to find the probability distribution of the levels with energy less than some E ($E<0; |E| \gg \mathscr{E}$), the distances between such states and the tunneling amplitude between them. Further we will work in a rather rough approximation similar to that used by Larkin and Ovchinnikov [31] and Imry and Ma [32]. However, we start with a rigorous statement of the problem which gives a clue for our further estimates. The most easily calculable value is the density of state $\nu(E,\Omega)$ which can be written as a path integral (see the cited works [4, 5, 6]):

$$\nu(E,\Omega) = \frac{1}{\Omega}\int \delta(E - E[U(\mathbf{x})])\,dW[U(\mathbf{x}),\Omega], \tag{5}$$

where $E[U(\mathbf{x})]$ is the spectrum of eigenvalues of the Schrödinger equation (3) in the volume Ω.

In a large 3d volume the states with energy $E \gg \mathscr{E}$ are delocalized, whereas the states with negative energy sufficiently large by modulus ($E<0$ and $|E| \gg \mathscr{E}$) are strongly localized. The threshold of localization is a positive energy of the order of $\mathscr{E}$ [33]. In the interval between $\mathscr{E}$ and $-\mathscr{E}$ the transition from the extended to strongly localized states proceeds. The latter are supported by rare fluctuations of the random potential, which form a potential well sufficiently deep to have the negative energy E as its only bound state. As it is clearly seen from equations (2,5), the main exponential factor in the density of state can be found by minimization of the integral $\int_\Omega U^2(\mathbf{x})\,d\Omega$ at a fixed value of the energy level $E[U(\mathbf{x})]$, which is a functional of the random potential $U(\mathbf{x})$. The latter can be determined as a minimum of the energy over the wave function:

$$E[U(\mathbf{x})] = \min_{\psi(\mathbf{x})} E[U(\mathbf{x}),\psi(\mathbf{x})] = \min_{\psi(\mathbf{x})} \int \left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{x})|^2 + U(\mathbf{x})|\psi(\mathbf{x})|\right. \tag{6}$$

Thus, we need to minimize a functional:

$$F[U(\mathbf{x}),\psi(\mathbf{x})] = \frac{1}{2}\int U^2(\mathbf{x})\,d\Omega - \lambda\int\left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{x})|^2 + U(\mathbf{x})|\psi(\mathbf{x})|^2\right. \tag{7}$$

over $\psi(\mathbf{x})$ and $U(\mathbf{x})$. Here λ is a Lagrangian factor. The minimization over $\psi(\mathbf{x})$ leads to Schrödinger equation (3), whereas the minimization over $U(\mathbf{x})$ results in a relation between $U(\mathbf{x})$ and $\psi(\mathbf{x})$:

$$U(\mathbf{x}) = \lambda|\psi(\mathbf{x})|^2 \tag{8}$$

Thus, equation (3) turns into the Ginzburg-Landau equation. For our purpose the most important consequence of the relationship (8) is that the fluctuation potential well $U(\mathbf{x})$ has the same characteristic linear size R as the wave function $\psi(\mathbf{x})$. It is clear that the maximum probability requires the bound state with the fixed energy E to be the only bound state in the potential well. Otherwise, at the same energy, we need a deeper well, i.e. larger $U^2(\mathbf{x})$. For the same reason the fluctuation well must have the spherical shape. Let the radius of the well is R. Then the depth of the well can be estimated as $U_{\min} \sim -\frac{\hbar^2}{mR^2}$ and the energy level in it differs by a factor about 1/2: $E \sim -\frac{\hbar^2}{2mR^2}$. The exponential factor in the density of state reads:

$$\exp\left(-\frac{\frac{4\pi}{3}R^3U^2}{2\kappa^2}\right) = \exp\left(-\frac{\mathscr{L}}{R}\right) = \exp\left[-\left(\frac{|E|}{\mathscr{E}}\right)^{1/2}\right], \tag{9}$$

The result (9) is valid provided the number in the exponent is large, i.e. $R \ll \mathscr{L}$ and $|E| \gg \mathscr{E}$. Let us introduce the spatial density $n_w(E)$ with the energy less than E. It is related to the DOS by equation $n_w(E) = \int_{-\infty}^{E} \nu(E)dE$. For deep levels it can be also considered as the spatial density of states $n_w(R)$ with the radius less than R, where $R = \hbar/\sqrt{2m|E|}$. For such states $n_w(R)$ is proportional to a small exponent $\exp(-\sqrt{|E|/\mathscr{E}}) = \exp(-\mathscr{L}/R)$. From the dimensionality consideration it follows:

$$n_w(E) = R^{-3} f\left(\frac{\mathscr{L}}{R}\right) \exp\left(-\frac{\mathscr{L}}{R}\right). \tag{10}$$

The function $f(x)$ can be found from Ref.[34] to be proportional to $f(x) \sim x^{\alpha}$ with $\alpha = 1$. It will be inessential for further calculation. The average distance $d(R)$ between the wells of the radius less than R reads:

$$d(R) = n_w^{-1/3} = R f^{-1/3} \exp\left(\frac{\mathscr{L}}{3R}\right). \tag{11}$$

Thus, the distances between the wells are significantly larger than their sizes. The tunneling factor $t(R)$ between two typical wells with the radius R of the same order of magnitude is given by a semiclassical expression $t(R) = \exp\left(-\frac{1}{\hbar}\int |p|\, dl\right)$, where the path of integration connects the two wells. By the order of magnitude $p \sim \hbar/R$ and the length of the integration path is $\sim d(R)$. Therefore, $\frac{1}{\hbar}\int |p|\, dl \approx d/R \approx f^{-1/3}\exp\left(\frac{\mathscr{L}}{3R}\right)$ and

$$t(R) = \exp\left[-f^{-1/3}\exp\left(\frac{\mathscr{L}}{3R}\right)\right]. \tag{12}$$

At $R \sim \mathscr{L}/3$ or $E \sim -9\mathscr{E}$, the distances between the optimal potential wells become of the same order of magnitude as their size R. Simultaneously the tunneling amplitude between the wells becomes of the order of 1. The potential wells percolate and tunneling is not small, but the states still are not propagating due to the Anderson localization [1].

BOSE GAS IN A LARGE BOX WITH AN UNCORRELATED RANDOM POTENTIAL

In the ground state of an ideal Bose gas in a large box with the Gaussian random potential all particles are located at the deepest fluctuation level. In the box of cubic shape with the side L the deepest level which occurs with probability of the order of 1 has the radius R determined by equation: $L^3 n_w(R) = 1$, i.e. $R \sim \frac{\mathscr{L}}{3\ln(L/\mathscr{L})}$. The prefactor f introduces a negligible correction to the denominator of the order of $\ln\left(\ln\frac{L}{\mathscr{L}}\right)$. The corresponding energy is $E \sim -9\mathscr{E}\left(\ln\frac{L}{\mathscr{L}}\right)^2$. As we already mentioned such a state is *non-ergodic* since the location and the depth of the deepest level strongly depends on a specific realization of the disordered potential. Therefore, the average energy per particle and other properties averaged over the ensemble has nothing in common with the properties of a specific sample. Even an infinitely small repulsion makes the system ergodic in the thermodynamic limit, i.e. when first the size of the system grows to infinity and then the interaction goes to zero. In a sufficiently large volume any physical value per particle coincides with its average over the ensemble. The reason of such a sharp change is that, at any small but finite interaction, the energy of particles repulsion overcomes their attraction to the potential well when the number of particles increases. They will be redistributed over multiple wells. Since the distribution of wells in different parts of sufficiently large volume passes all possible random configurations with proper ensemble probabilities, the ergodicity is established. Below we find how the interacting particles eventually fill localized states. In a real experiment the Bose gas may be quenched in a metastable state depending on the cooling rate and other non-thermodynamic factors. This is what M.P.A. Fisher *et al.* [35] call the Bose glass. Such a state is also possible in the case of weakly repulsive Bose gas. However, as it will be demonstrated later, in the case of cooled alkali atoms the tunneling amplitude still remains large enough to ensure the relaxation to the equilibrium state in $10^{-3} \div 10^{-2} s$. Our further estimates relate to the real ground state.

As in the Bogoliubov's theory [36] and its extension by Belyaev [37], we assume that the gas criterion $na^3 \ll 1$ is satisfied. Here $n = N/\Omega$ is the average particle density; N is their total number and a is the scattering length. Implicitly our considerations takes in account the change of the optimal potential well due to the interaction.

Let the Bose gas with the average density of particles n fill all potential wells with the radii less than R in the ground state. The average number of particles per well is

$$N_w(R) = n/n_w(R). \tag{13}$$

The local density inside the well of the linear size R is

$$n_p(R) = \frac{3N_w(R)}{4\pi R^3}. \tag{14}$$

The gain of energy per particle due to random potential is $E(R) = -\frac{\hbar^2}{2mR^2}$; the repulsion energy due to interaction is equal to $gn_p(R) = \frac{3\hbar^2 N_w(R)a}{mR^3}$, where we used the well-known relation for an effective potential field induced by a gas of scatterers [38]. Minimizing the total energy per particle $E_{tot}(R) = -\frac{\hbar^2}{2mR^2} + \frac{3\hbar^2 N_w(R)a}{mR^3}$ over R we find the value of R corresponding to the minimum of energy at fixed n with the logarithmic precision:

$$R(n) = \frac{\mathscr{L}}{\ln(n_c/n)} \ . \tag{15}$$

where the critical density n_c is defined as follows:

$$n_c = \left(3\mathscr{L}^2 a\right)^{-1} \tag{16}$$

The factor f in equation (10) leads to corrections of the type $\ln(\ln(n_c/n))$ which can be neglected. Further we put $f = 1$. The average distance between the closest filled wells according to the corresponding expression $d(R)$ for single-particle states reads

$$d(n) = \mathscr{L}(\ln(n_c/n))^{-1}(n_c/n)^{1/3}. \tag{17}$$

It strongly exceeds the average size of the potential well (15) at $n \ll n_c$. At the same condition the chemical potential of atoms can be estimated as

$$\mu(n) = -\frac{\hbar^2}{2mR^2(n)} = -\frac{\mathscr{E}}{2}\left(\ln\frac{n_c}{n}\right)^2. \tag{18}$$

The tunneling amplitude $t(n)$ between two wells separated by a typical distance $d(n)$ can be found by employing the single particle result (12):

$$t(n) = \exp\left[-(n_c/n)^{1/3}\right]. \tag{19}$$

Thus, the Bose gas at $n \ll n_c$ is fragmented into multiple clusters of small size $R(n)$ separated by much larger distances $d(n)$ and containing about $\mathscr{L}/\left[3a\left(\ln\frac{n_c}{n}\right)^3\right]$ particles each. The amplitude of tunneling between the wells depends on the scattering length in a non-analytic way and is exponentially small for weak interaction. Therefore, the number of particles in each cluster is well defined. As a consequence, the phase is completely uncertain. Such a state is a singlet with non-uniformly distributed particles, a random singlet: the ground state is non-degenerate. The compressibility $\frac{\partial n}{\partial \mu} = \frac{n}{\mathscr{E}}\ln\frac{n_c}{n}$ is finite as expected for the Bose glass phase [35].

BOSONS IN ATOMIC TRAPS

Our results can be easily extended to bosons in a harmonic traps characterized by a potential

$$V_{trap} = \frac{m\omega^2 R^2}{2} = \frac{\hbar^2}{2m}\frac{R^2}{\ell^4} \tag{20}$$

where we introduced the oscillator length $\ell = \sqrt{\hbar/(m\omega)}$. The energy of the bosons includes now four terms: the kinetic energy, the confining potential energy of the trap, the repulsion from other particles and the energy of the random potential. Two of them, the interaction with the trap and the random potential tend to confine and localize the particle. The minimization of energy leads to four different regimes (see Figure 1).

1. *Weak disorder and weak interaction*: $3Na \ll \ell \ll \mathscr{L}$.

In this case the interaction can be neglected. Minimizing the remaining terms, the kinetic energy and energy of the trap, we find $R = \ell$. Physically it means that all particles are condensed in the oscillator ground state.

2. *Weak disorder and strong interaction:* $\ell \ll \mathscr{L}$, $\ell \ll 3Na$.

Neglecting the kinetic energy and minimizing remaining energy of traps plus the repulsion energy, one finds the result known as Thomas-Fermi approximation[13]: $R = \left(\frac{9}{2}Na\ell^4\right)^{1/5}$.

3. *Strong disorder and weak interaction :* $3Na \ll \mathscr{L} \ll \ell$.

In this range of variables the non-ergodic phase is realized. Since interaction is negligible, the particles find a random potential well with the deepest level and fall into it. Let such a well can be found at a distance $\sim L$ from the trap center. Its depth typically is about $9\mathscr{E}\ln^2(L/\mathscr{L})$. This gain of energy must be not less than the loss of the trap energy $m\omega^2 L^2/2$. A typical value of L appears when both this energies have the same order of magnitude. Thus, $L \approx 6\sqrt{2}\left(\ell^2/\mathscr{L}\right)\ln(\ell/\mathscr{L})$. A typical size of the well is $R \approx \mathscr{L}/(6\ln(\ell/\mathscr{L}))$.

4. *Strong disorder and moderate interaction:* $\mathscr{L} \ll 3Na \ll \ell$.

In this case the ergodicity is restored. Our experience with the gas in a box prompts that the gas cloud is split into fragments each occupying a random potential well from very small size till same size R depending on N. The typical disorder energy per particle is $\mu = -\mathscr{E}\left(\ln\frac{n_c}{n}\right)^2$. It becomes equal to the trap energy at the distance $L \sim \left(\ell^2/\mathscr{L}\right)\ln\Gamma$ where Γ is a new dimensionless parameter

$$\Gamma = \frac{\ell^6}{3Na\mathscr{L}^5}. \tag{21}$$

It is equal to the ratio n_c/n, where $n \sim N/L^3 \sim N\mathscr{L}^3/\ell^6$ is the average density. The state of the Bose gas is fragmented and strongly localized when Γ is large; the transition to delocalized superfluid state proceeds when this

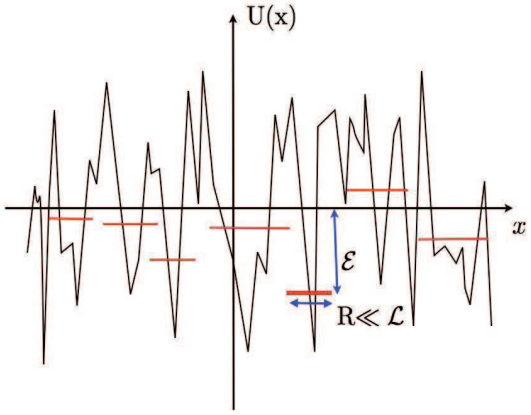

Figure 1. Regime diagram of atoms in traps: uncorrelated disorder. R denotes the size of the single existing atomic cloud. L is the size of the cloud of fragments.

ratio becomes $\sim$1. The phase diagram is shown in Fig. 1. Note the counter-intuitive dependence of the size on the number of particles: the cloud slightly contracts with increasing number of particles. It happens because the number of particles in each fragment increases more rapidly with the average density than the number of fragments.

CORRELATED DISORDER

So far we considered uncorrelated disorder (1). Our results can be extended to Gaussian random potentials with a finite correlation length b. The random potential is characterized by the correlation function $K(\mathbf{x},\mathbf{x}') = \langle U(\mathbf{x})U(\mathbf{x}')\rangle$ which has a form:

$$K(\mathbf{x},\mathbf{x}') = \langle U^2\rangle h\left(\frac{|\mathbf{x}-\mathbf{x}'|}{b}\right), \qquad (22)$$

where $\langle U^2\rangle \equiv \langle U^2(\mathbf{x})\rangle$ is the average quadratic fluctuation of the random field at a point. The shape function $h(u)$ is normalized by the condition $h(0)=1$. As long as $b \ll\leq \mathscr{L} = \frac{3\hbar^4}{4\pi m^2 U_0^2 b^3}$ the results of the previous considerations remain correct. In the opposite case the optimal potential well containing a deep level with negative energy E ($|E| \gg U_0 = \sqrt{\langle U^2\rangle}$) has the width b its shape coincides with that of the correlation function:

$$U(\mathbf{x}) = Eh\left(\frac{\mathbf{x}}{b}\right). \qquad (23)$$

Contrary to the uncorrelated case, it contains many bound states. The density of single-particle states is determined by minimization of the exponent in the Gaussian distribution

$$-\ln W[U(\mathbf{x})] = \frac{1}{2}\int U(\mathbf{x})K^{-1}(\mathbf{x},\mathbf{x}')U(\mathbf{x}')d\mathbf{x}d\mathbf{x}' \qquad (24)$$

over $U(\mathbf{x})$ at a fixed energy level in the fluctuation potential well $E[U(\mathbf{x})] = E$. Here $K^{-1}(\mathbf{x},\mathbf{x}')$ is the inverse correlator whose convolution with the correlator $K(\mathbf{x},\mathbf{x}')$ is equal to $\delta(\mathbf{x},\mathbf{x}')$. Substituting in eq. (24) $U(\mathbf{x})$ from eq. (23), we arrive at the Gaussian density of state and probability to find a deeply localized level with energy E:

$$q(E) = \exp\left(-\frac{E^2}{2}U_0^2\right) \qquad (25)$$

This Gaussian-like distribution over the energy was first found in the works [39, 40]; independently but later it was rediscovered in [41, 42]; several other works clarified and added some details to the theory [43, 44, 45] Again we briefly reproduce and extend their ideas to apply them for our problem. It is convenient to introduce a new length scale $B = b(\mathscr{L}/b)^{1/4} = (3/4\pi)^{1/4}\left(\hbar^2/mU_0\right)^{1/2}$. In the following we restrict our consideration to the case $b \gg \mathscr{L}$, i.e. $b \gg B$. Some results for uncorrelated disorder can be reproduced in the case of the strongly correlated disorder simply by substitution the Larkin length $\mathscr{L}$ by B. For example, the critical density is $n_c \sim 1/(aB^2)$. The deep potential wells are filled up to the depth $\sim U_0\sqrt{2\ln\frac{n_c}{n}}$; each well is filled only near the bottom, so that the size of a separate cloud (fragment) is $R \approx b(\ln(n_c/n))^{-1/2}$. The number of the particles in each fragment is $\mathscr{N} \approx \frac{4}{3}\pi R^3 n_c \sim \frac{b^3}{B^2 a}\left(\ln\frac{n_c}{n}\right)^{3/2}$. The distance between fragments is $d(n) \approx b(n_c/n)^{1/3}$.

In the case of a harmonic trap we again find four different regimes (Fig. 2): weak disorder and weak interaction; Thomas-Fermi regime; strong disorder and weak interaction (non-ergodic regime with one or few fragment); strong disorder and intermediate interaction (deeply localized fragmented state). The relevant parameter is $\Gamma' = \ell^6/(NaB^5) \approx n_c/n$. For $\Gamma' \approx 1$ the transition from deeply localized to superfluid phase proceeds.

LOWER DIMENSIONS

In the experiments with cooled gases the trap is mostly realized as a rotation ellipsoid with a large aspect ratio. In the experimental setup with He on Vycor experimenters used thin helium film, which displayed the crossover from 2d to 3d behavior. Therefore it is reasonable to extend our theory for lower dimensions $d = 1, 2$. Below we show the results without derivation, which can be found in our article [46].

The uncorrelated disorder is characterized by one only characteristic length, the Larkin length

$$\mathscr{L}_d = \left(\frac{\hbar^4}{m^2\kappa^2}\right)^{\frac{1}{4-d}}. \tag{26}$$

The characteristic energy scale is $\mathscr{E} = \hbar^2/(2m\mathscr{L}^2)$. The extension of the exponential law (9) for the probability to find the deep potential well of the size not larger than R is:

$$q(R) = f\exp\left[-\left(\frac{\mathscr{L}_d}{R}\right)^{4-d}\right] = f\exp\left[-\left(\frac{|E|}{\mathscr{E}_d}\right)^{\frac{4-d}{2}}\right], \tag{27}$$

As earlier f is a power-like function of the ratio $\mathscr{L}/R$ at $\mathscr{L}/R \gg 1$: $f(x) \propto x^{\alpha}$ with $\alpha = 1$ for $d = 2, 3$ and $\alpha = 0$ for $d = 1$. The average distance between the wells of the radius less than R is

$$d(R) = Rf^{-1/d}\exp\left[\frac{1}{d}\left(\frac{\mathscr{L}_d}{R}\right)^{4-d}\right] \tag{28}$$

The tunneling amplitude between nearest wells has a characteristic value

$$t(R) = \exp\left[-f^{-1/d}\exp\left[\frac{1}{d}\left(\frac{\mathscr{L}_d}{R}\right)^{4-d}\right]\right] \tag{29}$$

The repulsion energy per particle is as before $g_d n_p(R)$, but the connection between the coupling constant g and the scattering amplitude a depends on dimensionality of the space:

$$g_d = \frac{4\pi\hbar^2 a_d^{d-2}}{m} \tag{30}$$

At $d = 2$ this equation is invalid, but the coupling constant g is well defined and has dimensionality energy$\times$length^2. The maximal radius of the filled well as a function of density is

$$R(n) = \frac{\mathscr{L}_d}{\left(\ln\frac{n_c}{n}\right)^{\frac{1}{4-d}}}, \tag{31}$$

where the critical density is defined as follows:

$$n_c \sim \frac{\mathscr{E}_d}{g_d(4-d)} \sim \frac{1}{4\pi\mathscr{L}_d^2 a_d^{d-2}(4-d)} \tag{32}$$

In 1-d Bose-gas the critical density increases with the repulsion strength in contrast to 2- and 3-dimensional cases. The reason is that the 1-d Bose-gas with strong repulsion (hard core) is equivalent to the Fermi-gas whose kinetic energy increases with the size of the core. The chemical potential in the fragmented state reads $\mu(n) = -\mathscr{E}_d\left(\ln\frac{n_c}{n}\right)^{\frac{2}{4-d}}$. The tunneling amplitude between fragments is $t(n) = \exp\left[-\left(f\frac{n_c}{n}\right)^{1/d}\right]$. We do not show the results for strongly correlated potential in d dimensions. They can be found in the same article [46].

Considering the gas in a trap which has a shape of a disc or a cigar, one should express the parameters of effective $d-$dimensional problem in terms of the initial gas density n, 3d-scattering length a, Larkin length $\mathscr{L}$, the transverse oscillator length $\ell_\perp$ and the longitudinal oscillator length ℓ. The corresponding "translation rules" with precision of numerical factors are as follows:

- Larkin length: $\mathscr{L}_d \approx \left(\mathscr{L}\ell_\perp^{3-d}\right)^{\frac{1}{4-d}}$
- Coupling constant: $g_d = \frac{g\ell_\perp^{d-3}}{\Omega_{3-d}}$
- Scattering length: $a_d \sim \left(a\,\ell_\perp^{d-3}\right)^{\frac{1}{d-2}}$. This relation is invalid at $d = 2$, but the previous relation remains correct.
- Density: $n_d = n\ell_\perp^{3-d}$

CONCLUSIONS

Four parameters can be controllably and independently varied in the experiment with cooled gases. They are: number of particles N; the frequency ω or equivalently the strength of the trap; the scattering length a (it can be varied by approaching one of the Feshbach resonances); the strength of disorder U_0. Using this freedom it is feasible to pass all regimes described above. A simple estimate shows that, at $b \sim 1\mu m$, the transition from uncorrelated to strongly correlated regime proceeds at frequency of disorder potential $\omega_d = \sqrt{2U_0/mb^2} \sim 1kHz$ which is accessible.

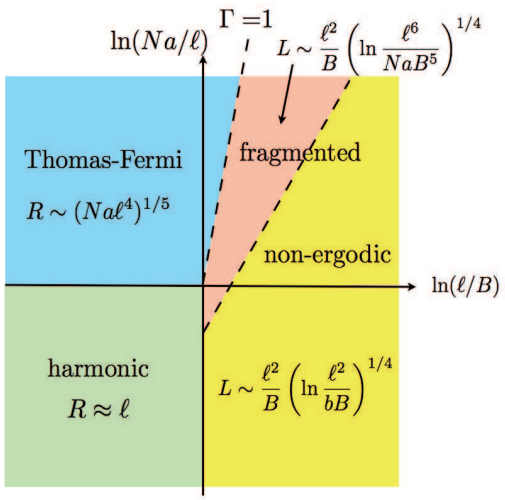

Figure 2. Regime diagram of atoms in traps: correlated disorder. R denotes the size of the single atomic cloud, L denotes the size of the fragmented state.

Simplest experiments are the measurements of the cloud size L as a function of different variable parameters in the regime of multiple localized fragments. Theory predicts that for uncorrelated disorder the size of the cloud is proportional to U_0^2/ω . It also predicts very weak dependence of the size on the number of particles $\sim \ln N$. In the case of strongly correlated disorder the size of the cloud is proportional to $\omega U_0^{1/2}$; the dependence on N also is weaker than in the uncorrelated regime: $L \propto (\ln N)^{1/4}$.

It would be important to observe the transition from non-ergodic state with one or few fragments to the ergodic state with many fragments and check that it happens at $N = \mathscr{L}/3a$ for uncorrelated disorder and at $N = \left(b^3/3aB^2\right)$ for strongly correlated disorder.

Another feasible experiment is the time-of-flight spectroscopy after switching off both the trap and the random potential. In this experiment the distribution of particles over momenta (velocities) is measured. Its width Δp is associated with the average size of the fragment R by the uncertainty relation $\Delta p = \hbar/R$. It gives the opportunity to check the equation $R = \mathscr{L}/\ln\Gamma$ for the uncorrelated disorder or $R = \mathscr{L}/\ln\Gamma'$ for correlated disorder. Installing a counter close to the trap, at a distance comparable to the size of the trap, would allow to register the oscillations of the particle flux due to discrete character of the fragmented state. This is an opportunity to find the distances between fragments and compare theory with experiment.

The transition between localized and delocalized coherent state in the random potential was found in several experiments (see Introduction). We propose to make more detailed measurement of the transition manifold and check our predictions.

An important question is whether the relaxation to the ground state can be reached during a reasonable time interval compatible with the time of experiment. We analyze this question for the uncorrelated or weakly correlated disorder. In this case the relaxation time due to tunneling can be estimated as $\tau = 2\pi\omega_n^{-1}t^{-1}$, where $\omega_n \sim \frac{\mathscr{E}}{\hbar}(\ln\Gamma)^2$ is the characteristic frequency of the optimal potential well and $t \sim \exp\left[-\Gamma^{1/3}\right]$ is the tunneling coefficient (see eq. (19)). For numerical estimates we accept $\Gamma \sim 125$, $\ell \sim 10\mu m$, $b \sim \mathscr{L} \sim 1\mu m$, $a \sim 0.01\mu m$, $N \simeq 27{,}000$. Then $t^{-1} = 148$ and $\tau \sim 0.06s$. The Larkin length can be increased by decreasing the amplitude of the random potential. Simultaneously, at fixed values N, ℓ and a, the value Γ decreases as $\mathscr{L}^{-5}$. This example shows that the equilibrium is accessible, though it is difficult to reach large ratio $\mathscr{L}/b$.

In conclusion, we presented a simple physical picture of deeply localized states of the Bose gas in a random potential. We demonstrated that the particles eventually fill the deep potential wells formed by fluctuations of the random potential and by their self-consistent field. Based on this idea the geometrical and physical properties of these states are described. It occurs that the ground state of the system can be either almost homogeneous and coherent (superfluid) if the disorder is weak enough, or fragmented and strongly localized. In particular, if the disorder is much stronger than the repulsion between particles, the system is in the non-ergodic state, which properties even in the equilibrium strongly depend on the specific sample. At growing number of particles the system occurs in an ergodic, but strongly localized ground state consisting of multiple particle clusters populating deep fluctuation wells. At the further increase of the number of particles the tunneling between different potential wells

increases leading to the phase correlation and finally to the quantum phase transition to the coherent (superfluid) state.

Quite recently there appeared a work by A. Babichenko and V. Babichenko [47] in which the authors have formulated the problem of deeply localized state of weakly interacting Bose gas employing the Keldysh-Schwinger technique. For the uncorrelated Gaussian disorder they have found the instanton solution corresponding to the deeply localized state at a fixed chemical potential and the expression for the chemical potential vs. given average density n, which coincides with our Eq. (18). Thus, they confirmed our theory by exact calculation.

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