One-dimensional Anderson Localization: distribution of wavefunction amplitude and phase at the band center.

V. E. Kravtsov* and V. I. Yudson†

*The Abdus Salam International Centre for Theoretical Physics, P.O.B. 586, 34100 Trieste, Italy.
[2]Landau Institute for Theoretical Physics, 2 Kosygina st.,117940 Moscow, Russia
†Institute for Spectroscopy, Russian Academy of Sciences, 142190 Troitsk, Moscow reg., Russia.

Abstract. The statistics of normalized wavefunctions in the one-dimensional (1d) Anderson model of localization is considered. It is shown that at any energy that corresponds to a rational filling factor $f = \frac{p}{q}$ there is a statistical anomaly which is seen in expansion of the generating function (GF) to the order $q-2$ in the disorder parameter. We study in detail the principle anomaly at $f = \frac{1}{2}$ that appears in the leading order. The transfer-matrix equation of the Fokker-Planck type with a two-dimensional internal space is derived for GF. It is shown that the zero-mode variant of this equation is integrable and a solution for the generating function is found in the thermodynamic limit.

Keywords: localization, mesoscopic fluctuations
PACS: 72.15.Rn, 72.70.+m, 72.20.Ht, 73.23.-b

INTRODUCTION

Anderson localization (AL) enjoys an unusual fate of being a subject of advanced research during a half of century. The seminal paper by P.W.Anderson [1] opened up a direction of research on the interplay of quantum mechanics and disorder which is of fundamental interest up to now [2]. The one-dimensional tight-binding model with diagonal disorder –the Anderson model (AM)– which is the simplest and the most studied model of this type, became a paradigm of AL:

$$\hat{H} = \sum_i \varepsilon_i c_i^\dagger c_i - \sum_i t_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right). \qquad (1)$$

In this model the hopping integral is deterministic $t_i = t$ and the on-site energy ε_i is a random Gaussian variable uncorrelated at different sites and characterized by the variance $\langle(\delta\varepsilon_i)^2\rangle = w$. The dimensionless parameter $\alpha^2 = w/t^2$ describes the strength of disorder.

The best studied is the continuous limit of this model in which the lattice constant $a \to 0$ at ta^2 remaining finite [3, 4, 5, 6]. There was also a great deal of activity [8, 7] aimed at a rigorous mathematical description of 1d AL. However, despite considerable efforts invested, some subtle issues concerning 1d AM still remain unsolved. One of them is the effects of commensurability between the de-Broglie wavelength λ_E (which depends on the energy E) and the lattice constant a. The parameter that controls the commensurability effects is the filling factor $f = \frac{2a}{\lambda_E}$ (fraction of states below the energy E).

It was known for quite a while [9, 10] that the Lyapunov exponent takes anomalous values at the filling factors equal to $\frac{1}{2}$ and $\frac{1}{3}$ (compared to those at filling factors f beyond the window of the size $\alpha^2 \ll 1$ around $f = \frac{1}{2}$ and $f = \frac{1}{3}$). At weak disorder the Lyapunov exponent sharply *decreases* at $f = \frac{1}{2}$ (which is usually associated with *increasing* the localization length). However, near $f = \frac{1}{3}$ the Lyapunov exponent exhibits a sharp *peak* if the third moment of the on-site energy distribution is non-zero [10]. More recently [12, 13] it was found that the statistics of conductance in 1d AM is anomalous at the center of the band that corresponds to the filling factor $f = \frac{1}{2}$. We want to stress that all these anomalies were observed for the AM Eq.(1) in which the on-site energy ε_i is random. This Hamiltonian does not possess the *chiral symmetry* [14, 2] which is behind the statistical anomalies at the center of the band $E = 0$ in the *Lifshitz model* described by Eq.(1) with the deterministic $\varepsilon_i = 0$ and a random hopping integral t_i. Thus the statistical anomaly at $f = \frac{1}{2}$ raises a question about a *hidden symmetry* that do not merely reduce to the two-sublattice division [14, 13, 2].

The above results point out to existence of the entire *devil's staircase* of statistical anomalies at a rational filling factor f in different physical quantities of 1d Anderson model. There are numerous questions concerning physics behind the anomalies. One of such puzzles is the sign of the variation of the Lyapunov exponent which corresponds to suppression of localization in the vicinity of $f = \frac{1}{2}$ and to enhancement of localization near $f = \frac{1}{3}$ (provided that there is an asymmetry of the on-site energy distribution). It is difficult to explain such behavior by a conventional model invoking reflection off Bragg mirrors with doubling or tripling the lattice period due to peculiar fluctuations of the random landscape of ε_i [15], as this model seems to favor localization in all cases.

A possible resolution of this conflict between physical intuition and mathematical results is that the Lyapunov exponent describes only the tails of the localized wavefunction while the global picture of localization is better represented by the average *inverse participation ratio* $I = \int dx \, \langle |\psi(x)|^4 \rangle$, where $\psi(x)$ is a random *normalized* eigenfunction obeying the Shroedinger equation $\hat{H}\psi = E_n \psi$ and the boundary conditions on both ends of the chain. Here it is worth noting that computing of I is much more difficult than the problem of Lyapunov exponent, since the latter does not require an *eigenfunction* of the Shroedinger operator obeying all the boundary conditions, while the inverse participation ratio is defined only for *normalizible eigenfunctions*.

In this paper we give a regular description of the phenomenon of the rational-f statistical anomalies in terms of the generalized transfer-matrix equation (TME) which is a universal tool to describe properties of a generic 1d or quasi-1d system. As the result we obtain the *joint probability distribution* (JPD) $\mathscr{P}(u,\phi)$ of the amplitude u and phase ϕ of the random eigenfunction $\psi \sim \sqrt{u}\cos\phi$ which can be used to compute *any* local statistics of *normalized* eigenfunctions of 1d AM.

We will show that the TME for $f = \frac{1}{2}$ has anomalous terms which make it essentially two-dimensional second-order PDE depending on the amplitude variable u as well as on the phase variable ϕ. Yet, we show that this equation has an exact solution for the zero mode and we find this solution explicitly in quadratures. Similar anomalous terms are shown to appear at any other rational filling factor $f = \frac{p}{q}$.

DERIVATION OF THE TM EQUATION.

The starting point of our analysis is the TM equation for the generating function (GF) $\Phi_j(u,\phi)$ on the lattice site j:

$$\Phi_{j+1}(u,\phi) = \left(1 + \frac{2a}{\ell_0}\left[\mathscr{L}(u,\phi) - c_1(\phi)\,u\right]\right)\Phi_j(u,\phi - \pi f), \tag{2}$$

where $\ell_0 = a\frac{2t^2}{w}\sin^2(\pi f)$ is the "bare" localization length; in the limit of weak disorder $\mathscr{L}(u,\phi) = c_2(\phi)\,u^2\partial_u^2 + c_3(\phi)\,(u\partial_u - 1) + c_4(\phi)\,u\partial_u\partial_\phi + c_5(\phi)\,\partial_\phi + c_6(\phi)\,\partial_\phi^2$. The coefficients $c_i(\phi)$ are all combinations of $\cos(2\phi)$ and $\sin(2\phi)$ which at first glance do not show any nice structure: $c_1(\phi) = \frac{1}{2}(1+\cos(2\phi))$, $c_2(\phi) = 1 - \cos^2(2\phi)$, $c_3(\phi) = -(1 - \cos(2\phi) - 2\cos^2(2\phi))$, $c_4(\phi) = \sin(2\phi)(1+\cos(2\phi))$, $c_5(\phi) = -\frac{3}{2}\sin(2\phi)(1+\cos(2\phi))$, $c_6(\phi) = \frac{1}{4}(1+\cos(2\phi))^2$. This equation has been derived in Ref.[16] by expansion to the first order in α^2 of the exact integral TME obtained by the super-symmetry method [17].

The GF $\Phi(u,\phi)$ determines the JPD $\mathscr{P}(u,\phi)$ of eigenfunction amplitude and phase *in the bulk* in a long chain $\mathscr{P}(u,\phi)$. However the relationship between them is non-trivial:

$$\mathscr{P}(u,\phi) = -\frac{1}{2\pi i u}\,\partial_u \int_{-i\infty+0}^{+i\infty+0} \frac{dy}{y}\, e^y\, \Phi^2(uy,\phi). \tag{3}$$

The main feature of Eq.(3) is that it is *quadratic* in $\Phi(u,\phi)$. This reflects the identical boundary conditions at the *two* ends of a chain [18] and the fact that the point of observation is in the bulk. In contrast to that in the problem of Lyapunov exponent one considers essentially a *semi-infinite* chain with points of observation close to its end. In this case the GF itself plays a role of the distribution function.

By construction [16] the function $\Phi_j(u,\phi)$ must be periodic in ϕ with the period of π which corresponds to the phase factor $\cos\phi$ of the wave function sweeping all possible values in the interval $[0,\pi]$. However, the shift in the argument ϕ in the r.h.s. of Eq.(2) is by a *fraction f* of π. For a rational $f = \frac{p}{q}$ one has to make q iterations in Eq.(2) in order to get a closed equation for the GF. In the leading order in α we obtain:

$$\Phi_{j+q}(u,\phi) - \Phi_j(u,\phi) = \frac{2a}{\ell_0} \times \left[\sum_{r=0}^{q-1}\mathscr{L}(\phi - r\pi p/q) - u\sum_{r=0}^{q-1} c_1(\phi - r\pi p/q)\right]\Phi_j(u,\phi). \tag{4}$$

The reason for the principle anomaly at $f = \frac{1}{2}$ is the following identity that shows a jump at $q = 2$:

$$\sum_{r=0}^{q-1} e^{2i\phi - 2ir\pi p/q} = 0, \quad \sum_{r=0}^{q-1} e^{4i\phi - 4ir\pi p/q} = \begin{cases} 0, & q > 2 \\ 2e^{4i\phi}, & q = 2 \end{cases} \tag{5}$$

Assuming $q \ll \ell_0/a$, expanding the l.h.s. of Eq.(4) and introducing the dimensionless coordinate $x = ja/\ell_0$ we obtain:

$$\partial_x\Phi = \left[u^2\partial_u^2 - u + \frac{3}{4}\partial_\phi^2\right]\Phi + \delta_{f,\frac{1}{2}}\,\Delta\mathscr{L}(u,\phi)\,\Phi. \tag{6}$$

The second term in Eq.(6) is the anomaly that is present only for the filling factor $f = \frac{1}{2}$. The corresponding operator takes the form:

$$\begin{aligned} \Delta\mathscr{L} &= \cos(4\phi)\left[-u^2\partial_u^2 + 2u\partial_u + \frac{1}{4}\partial_\phi^2 - 2\right] \\ &+ \sin(4\phi)\left[u\partial_u\partial_\phi - \frac{3}{2}\partial_\phi\right]. \end{aligned} \tag{7}$$

Without this part, the variables u and ϕ are separated and one can immediately find the independent of x solution $\Phi(u) = \sqrt{u}K_1(2\sqrt{u})$. This *zero mode* solution describes the limit of a long chain with the length $L \gg \ell_0$. It has been earlier obtained [6] in the continuous limit $f \ll 1$. It also arises in the theory of a multi-channel disordered wire [17, 18]. Plugging this solution into Eq.(3) we obtain the following probability distribution of $|\psi|^2$ in a long *strictly* one-dimensional system (amazingly, this result was not known before):

$$P(|\psi|^2) = \frac{\ell_0}{L} \frac{\exp\left(-|\psi|^2\ell_0\right)}{|\psi|^2}. \tag{8}$$

This distribution is valid for $|\psi|^2\ell_0 \gg e^{-L/\ell_0}$ and should be cut off at very small $|\psi|^2$ to ensure normalizability [19].

At $f = \frac{1}{2}$ the quantities u and ϕ are no longer independent. Furthermore, the integrability of Eq.(6) – even in its zero-mode variant– is not guaranteed. Yet, with a suitable choice of co-ordinates the variables are separated in the zero-mode TM equation.

SEPARATION OF VARIABLES.

The integrability of the zero mode TME Eq.(6) is shown in three steps. The step one is to introduce new set of variables u and $v = u\cos(2\phi)$ instead of (u,ϕ) and a new function $\tilde{\Phi}(u,v) = u^{-1}\,\Phi(u, \frac{1}{2}\arccos(v/u))$. In these variables the zero-mode TME Eq.(6) takes a very symmetric form:

$$[D_1^2 + D_3^2]\,\tilde{\Phi} = \frac{u}{2}\,\tilde{\Phi}, \tag{9}$$

where the operators D_1 and D_3 belong to the family of three operators from the representation of the sl_2 algebra:

$$D_{1(3)} = \pm\sqrt{u^2 - v^2}\,\partial_{u(v)},\; D_2 = u\partial_v + v\partial_u, \tag{10}$$

obeying the commutation relations:

$$[D_1, D_2] = -D_3,\; [D_3, D_1] = D_2,\; [D_2, D_3] = D_1. \tag{11}$$

Now it is clear that there is a hidden order in a set of ϕ-dependent terms in Eq.(7) and the way they match the regular part in r.h.s. of Eq.(6).

One can further extend the algebra including also the operator u in the r.h.s. of Eq.(9). To this end we define:

$$B_1 = v,\;\; B_2 = \sqrt{u^2 - v^2},\;\; B_3 = u. \tag{12}$$

One can easily check that

$$[B_i, B_j] = 0,\;\; [D_i, B_j] = e_{ijk} B_k. \tag{13}$$

The 6-dimensional algebra defined by Eqs.(11),(13) constitutes the closed set of operators sufficient to formulate all the symmetries of Eq.(9). Establishing the symmetries and the corresponding operators commuting with the "Hamiltonian" $D_1^2 + D_3^2 - \frac{1}{2}B_3$ is an important task which was not accomplished so far. When achieved, it would probably help to construct the new co-ordinates (very much in the same way as the Kramers symmetry for the 3D Coulomb problem helps to identify the set of parabolic co-ordinates) which would allow for a complete solution to the problem. However, for the time being we proceed with guessing the coordinates to separate variables in the zero-mode problem.

The next step is to transform Eq.(9) to the Schroedinger-like equation $H\Psi \equiv -(\partial_u^2 + \partial_v^2)\Psi + U(u,v)\Psi = 0$ for the function $\Psi(u,v) = (u^2 - v^2)^{\frac{1}{4}}\,\tilde{\Phi}$, where

$$U = -\frac{3}{4}\frac{u^2 + v^2}{(u^2 - v^2)^2} + \frac{1}{2}\frac{u}{u^2 - v^2}. \tag{14}$$

Finally we introduce the variables

$$\xi = \frac{u + v}{2} = u\cos^2\phi,\quad \eta = \frac{u - v}{2} = u\sin^2\phi. \tag{15}$$

It is easy to see that in these variables the operator in Eq.(14) becomes a sum of two identical *one-dimensional* Hamiltonians $H = \hat{H}_\xi + \hat{H}_\eta$ where $\hat{H}_\xi$ is given by:

$$\hat{H}_\xi = -\partial_\xi^2 - \frac{3}{16}\frac{1}{\xi^2} + \frac{1}{4\xi}. \tag{16}$$

Thus in new variables Eq.(15) the TME Eq.(6) is separable also at $f = \frac{1}{2}$ and can be reduced to the two ODE's of the Schredinger type $\hat{H}_\xi\,\varphi_\lambda(\xi) = \lambda\,\varphi_\lambda(\xi)$ and $\hat{H}_\eta\,\varphi_{-\lambda}(\eta) = -\lambda\,\varphi_{-\lambda}(\eta)$ which have a well-known solution in terms of the hypergeometric functions (Whittaker functions) [21].

Remarkably, ξ and η play a role of the co-ordinate and the momentum in the equivalent classical model of kicked oscillator [11].

UNIQUENESS OF THE SOLUTION.

The general solution to the "Schroedinger equation" $H\Psi = 0$ is given by the integral over the parameter λ:

$$\Psi = \int d\lambda d\bar{\lambda}\; c(\lambda,\bar{\lambda})\,\varphi_\lambda(\xi)\,\varphi_{-\lambda}(\eta), \tag{17}$$

where integration is generically over the complex plane of λ and $c(\lambda,\bar{\lambda})$ is an arbitrary function [22]. How does this huge degeneracy comply with the intuitive expectation that the statistics of wavefunctions in an infinite disordered chain should be unique and independent of the boundary conditions? Below we show that the natural physical requirements on $\Phi(u,\phi)$ help to determine GF up to a constant factor which can be further fixed using the wave function normalization $\langle|\psi|^2\rangle = \frac{1}{L}$.

First of all we note that $F(\lambda;\xi,\eta)=\varphi_\lambda(\xi)\,\varphi_{-\lambda}(\eta)$ is a holomorphic function of λ, i.e. it depends only on $\lambda=\rho e^{i\sigma}$ but not on $\bar{\lambda}=\rho e^{-i\sigma}$. The idea is to represent the integral over the complex plane as an integral over ρ and σ and then rotate the contour of integration $\rho\to te^{-i\sigma}$ so that the dependence on σ remains only in $c(\lambda,\bar{\lambda})$ and in the integration measure but not in $F(\lambda;\xi,\eta)$. Then performing integration over σ one obtains a new function $C(t)=t\int d\sigma\, e^{-2i\sigma}\,c(t,te^{-2i\sigma})$ which stands for $c(\lambda,\bar{\lambda})$ in an expression similar to Eq.(17) but involving only a one-dimensional *contour integral*. This contour can be further rotated to make the expression more symmetric. Thus without loss of generality we write a solution to the zero-mode TM equation Eq.(9) for $f=\frac{1}{2}$:

$$\Phi(\xi,\eta)=\frac{\xi+\eta}{(\xi\eta)^{1/4}}\int_0^\infty d\lambda\, C(\lambda) \tag{18}$$
$$\times\quad\left[W_{-\lambda\varepsilon,\frac{1}{4}}\left(\frac{\bar{\varepsilon}\xi}{4\lambda}\right)W_{-\lambda\bar{\varepsilon},\frac{1}{4}}\left(\frac{\varepsilon\eta}{4\lambda}\right)+c.c\right].$$

Here $W_{\kappa,\mu}(z)$ is the Whittaker function [21]; $\varepsilon=e^{i\pi/4}$, $\bar{\varepsilon}=e^{-i\pi/4}$, and $C(\lambda)$ is a real function yet to be determined.

Before we proceed with determining this function it is important to establish its properties as $\lambda\to 0$. To this end we note that $\langle|\psi|^2\rangle=\int d\phi\, du\, u\cos^2\phi\;\mathscr{P}(u,\phi)\propto\int d\phi\;\cos^2\phi\;\Phi^2(0,\phi)$. This is immediately seen upon integration by parts over u in Eq.(3). Thus the GF $\Phi(\xi,\eta)$ must tend to a finite limit as $\xi\to 0$ and $\eta\to 0$. Given the asymptotic behavior of Whittaker functions this is equivalent to:

$$C(\lambda)=\lambda^{-\frac{3}{2}}\,\tilde{C}(\lambda),\quad \tilde{C}(0)=\text{const}. \tag{19}$$

GF defined by Eq.(18) is periodic in ϕ with the period $\frac{\pi}{2}$ as it should be for $q=2$. This is guaranteed by the adding of the *c.c* term in Eq.(18). What is not automatically guaranteed is that $\Phi(\xi,\eta)$ is *smooth* as a function of ϕ at $\phi=0$. We will see that it is the requirement of *smoothness* at $\phi=0$ which fixes (up to a constant factor) the unknown function $\tilde{C}(\lambda)$.

Indeed, the discontinuity of derivatives at $\phi=0$ may arise from the branching of the expression in Eq.(18) at a small η. From the representation of the Whittaker function in terms of the hypergeometric functions one concludes that the general solution Eq.(18) is a sum of a part which is regular in the vicinity of $\eta=0$ and a part which has a square-root singularity $\sqrt{\eta}\approx\sqrt{u}|\phi|$. The condition that this latter part cancels out in the solution Eq.(18) is the following (t is real):

$$\Im\left[\frac{\tilde{C}(\bar{\varepsilon}t)}{\Gamma\left(\frac{1}{4}-it\right)}\,e^{-\frac{i\eta}{8t}}\,{}_1F_1\left(\frac{3}{4}-it,\frac{3}{2},\frac{i\eta}{4t}\right)\right]=0 \tag{20}$$

The crucial fact for the possibility to fulfil this condition is the identity for the hypergeometric functions [21]:

$$e^{-z/2}\,{}_1F_1\left(\frac{3}{4}-it,\frac{3}{2},z\right)=e^{z/2}\,{}_1F_1\left(\frac{3}{4}+it,\frac{3}{2},-z\right). \tag{21}$$

Now one can immediately guess the solution for $\tilde{C}(\lambda)$:

$$C_0(\lambda)=\Gamma\left(\frac{1}{4}+\varepsilon\lambda\right)\Gamma\left(\frac{1}{4}+\bar{\varepsilon}\lambda\right). \tag{22}$$

It is easy to see that the general solution to Eq.(20)is

$$\tilde{C}(\lambda)=C_0(\lambda)S(\lambda)=C_0(\lambda)\sum_{k=0}^{\infty}a_k\lambda^{4k}, \tag{23}$$

where the function $S(\lambda)$ must be regular in the entire complex plane of λ. Now we apply the condition of convergence of the integral over λ in Eq.(18) at large λ to find the allowed asymptotic behavior of $S(\lambda)$ at $\lambda\to\infty$. Substituting Eq.(23) into Eq.(18) and using the asymptotics of the Whittaker and Γ-functions we find that the integrand behaves as $\lambda^{-3}S(\lambda)$ at $\lambda\to\infty$. This means that $|S(\lambda)|$ should increase not faster than λ^2. There is only one such entire function with the structure of Eq.(23): this is a constant $S(\lambda)=a_0=\text{const}$. This constant has to be determined from the normalization condition $\langle|\psi|^2\rangle=L^{-1}$ using Eq.(3)

CONCLUSION AND DISCUSSION

Eqs.(18),(19),(22) is the main result of the paper. They give an exact and unique solution for the generating function at $f=\frac{1}{2}$ anomaly. The latter determines the JPD of eigenfunction amplitude and phase, Eq.(3) which can be used to compute all local statistics of the one-dimensional Anderson model in the bulk of a long chain $L\gg\ell_0$. The integrability of TME Eq.(6) suggests that there is a hidden symmetry of the problem at $f=\frac{1}{2}$. We make a conjecture that this symmetry is naturally formulated in the three dimensional space rather than in the two-dimensional space (ξ,η) and that it has to do with the symmetry of the 3d harmonic oscillator. This conjecture is based on an analogy between our main result Eq.(18) and the expression for the Green's function of the 3d harmonic oscillator problem [23]. This analogy concerns the parameter (λ in our problem and k in Ref. [23]) entering both in the argument of the Whittaker functions and in its first index in a mutually reciprocal way, as well as the second index of the Whittaker functions being $\frac{1}{4}$ in both cases. Establishing this symmetry would also be useful for studying the anomalies at $f=\frac{p}{q}$ with $q>2$.

We have obtained [20] the anomalous operator $\Delta^{(3)}\mathscr{L}(u,\phi)$ which stands for $\Delta\mathscr{L}$ in Eq.(6) at $f=\frac{1}{3}$

and shown that the mechanism similar to Eq.(5) is also responsible for the anomaly at $f = \frac{1}{3}$. The results of this study will be published elsewhere.

ACKNOWLEDGMENTS

We appreciate stimulating discussions with A.Agrachev, Y.V.Fyodorov, A.Kamenev and A.Ossipov and a support from RFBR grant 06-02-16744.

REFERENCES

1. P. W. Anderson, *Phys. Rev.* **1**09, 1492 (1958).
2. F. Evers and A. D. Mirlin, *Anderson transition*, ArXiv:0707.4378 (2007).
3. V. L. Berezinskii, *Zh. Exp. Teor. Fiz.* **65**, 1251 (1973) [*Sov. Phys. JETP* **38**, 620 (1974)].
4. A. A. Abrikosov and I. A. Ryzhkin, *Adv. Phys.* **2**7, 147 (1978).
5. V. I. Melnikov, *JETP Lett.* **32**, 255 (1980).
6. I. V. Kolokolov, *Zh. Exp. Teor. Fiz.*__103__, 2196 (1993) [*JETP* **76**, 1099 (1993)].
7. I. M. Lifshitz, S. A. Gredeskul and L. A. Pastur *Introduction to the theory of disordered systems* (Wiley, New York, 1988).
8. J. Frohlich, F. Martinelli, E. Scoppola and T. Spencer, *Comm. Math. Phys.* **101**, 21 (1985).
9. M. Kappus and F. Wegner, *Z. Phys. B* **45**, 15 (1981).
10. B. Derrida and E. Gardner, *J. Phys. (Paris)* **45**, 1283 (1984).
11. L. Tessieri and F. M. Izrailev, *Phys. Rev. E* **62**, 3090 (2000).
12. H. Schomerus and M. Titov, *Phys. Rev. B* **67**, 100201(R) (2003).
13. L. I. Deych, et al. *Phys. Rev. Lett.* **91**, 096601 (2003).
14. F. J. Dyson, *Phys. Rev.* **92**, 1331 (1958).
15. V. M. Apalkov, M. E. Raikh and B. Shapiro, *Phys. Rev. Lett.* **92**, 066601 (2004).
16. A. Ossipov and V. E. Kravtsov, *Phys. Rev. B* **73**, 033105 (2006).
17. K. B. Efetov, *Supersymmetry in chaos and disorder* (Cambridge University Press, Cambridge, England, 1977).
18. A. D. Mirlin, *Phys. Rep.* **326**, 259 (2000).
19. All the positive moments of this distribution are finite, in particular $\langle|\psi|^2\rangle = \frac{1}{L}$.
20. In the next order of expansion in α^2 there appear anomalous terms at $f = \frac{1}{3}$ and $f = \frac{2}{3}$ proportional to $\sin(6\phi)$ and $\cos(6\phi)$ containing up to the third derivative wrt u and ϕ.
21. I. S. Gradshtein and I. M. Ryzhik *Table of integrals series and products* (Academic Press, New York, 1996).
22. Note that the "Hamiltonian" in Eq.(16) is a non-Hermitean operator. This is a consequence of the singular inverse-quare potential. To make it Hermitean one has to impose a condition $\varphi(0) = 0$ assuming a hard wall at $\xi < 0$. There is no such condition in our problem.
23. V. L. Bakhrakh and S. I. Vetchinkin, *Theor. Math. Phys.* **6**, 283 (1971)[*Sov. Phys: Teor. Mat. Fiz.* textbf 6,392 (1971)].