d-wave to *s*-wave to normal metal transitions in disordered superconductors

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Abstract. We study suppression of superconductivity by disorder in *d*-wave superconductors, and predict the existence of (at least) two sequential low temperature transitions as a function of increasing disorder: a *d*-wave to *s*-wave, and then an s-wave to metal transition. This is a universal property of the system which is independent of the sign of the interaction constant in the *s*-channel.

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Generally the order parameter in superconductors is a function of two coordinates and two spin indices. Classification of possible superconducting phases in crystalline materials was done in [1, 2]. A majority of low-T_c crystalline superconductors have a singlet order parameter with *s*-wave symmetry. It does not change its sign under rotation, and in the isotropic case can be approximated by a complex number $\Delta^s(\mathbf{r}) = \Delta(\mathbf{r},\mathbf{r})$. However, over the last decades a number of superconductors have been discovered in which the order parameter changes sign under rotation. A notable example is HTC superconductors, where in the absence of disorder the order parameter has singlet *d*-wave symmetry [4, 3]: $\Delta(\mathbf{r}-\mathbf{r}')$ changes sign under rotation by $\pi/2$, and consequently $\Delta(\mathbf{r},\mathbf{r}) = 0$. This means that the Fourier transform $\Delta(\mathbf{k})$ changes its sign under a $\pi/2$ rotation as well, as is shown schematically by the rosettes in Fig.1. Since the sign of $\Delta(\mathbf{k})$ in crystalline *d*-wave superconductors depends on the direction of the wave vector $\mathbf{k}$, they are much more sensitive to disorder than *s*-wave superconductors: at temperature $T = 0$, *d*-wave superconductivity gets destroyed when the electron mean free path l is of the order of the zero temperature coherence length in a pure superconductor, $l \sim l_0 = 1.78\xi_o \gg 1/k_F$. Here k_F is the Fermi wavelength. This is in contrast with the case of *s*-wave superconductors, where according to the Anderson theorem the superconductivity is destroyed at much higher level of disorder, when $l \sim 1/k_F$. The fate of the *d*-wave superconductors at $l < \xi_0$ depends on the sign of the interaction constant λ_s in the s-wave channel. If the interaction λ_s in the s-wave channel is attractive, but weaker than the attraction in the *d*-wave channel $|\lambda_s| < |\lambda_D|$, then at weak disorder, ($l > \xi_0$), the superconducting order parameter has d-wave symmetry, while at $l < \xi_0$ the disorder destroys the *d*-wave superconductivity and the system undergoes a phase transition into an s-wave superconducting state. (See, for example, [5]).

In this article we consider a more interesting case, in which the interaction in the *s*-channel is repulsive at strong enough disorder $1/k_F \ll l \ll \xi_0$ the system is in normal state. We predict at least two low- temperature phase transitions: a *d*-wave to *s*-wave, and then an *s*-wave to normal metal transition. Qualitatively the phase diagram of disordered *d*-wave superconductors is shown in Fig.1. Let us first discuss the definition of *s*- and *d*-symmetries in bulk disordered systems. Before averaging over random realizations of disorder, the system does not possess any particular spacial symmetry at all. However in bulk samples, the symmetry is restored upon configuration averaging. We can think of several different definitions of the global symmetry of the order parameter: a) An operational definition is provided by the result of a phase sensitive experiment, such as the corner SQUID experiment, for example, [3, 4]. b) The quantity $\overline{\Delta(\mathbf{r},\mathbf{r}')}$ can be characterized as having *d*-wave or *s*-wave symmetry. Here the over-line stands for the averaging over the sample volume. c) A globally *s*-wave component of the order parameter can be defined in terms of the local *s*-component of the anomalous Green function $\mathscr{F}(\mathbf{r}=\mathbf{r}') \equiv \mathscr{F}^{(s)}(\mathbf{r})$. If we define $P_{\pm}$ to be the volume fraction of a sample where $F^{(s)}(\mathbf{r})$ has a positive or negative sign, respectively, then the system has an s-wave component if $(P^+ - P^-) \neq 0$. These definitions may be not equivalent under all circumstances. However, for the most part, we will deal with the interval of parameters in which all these definitions are approximately interchangeable.

It is important to realize that it is inevitable near criticality to have a situation in which the local pairing in disordered superconductors is "*d*-wavelike" and yet the global superconductivity has s-wave symmetry. The *d*-wave to *s*-wave transition can be understood at the mean

field level. The electron mean free path is an average characteristic of disorder. Let us introduce a "local" value of the mean free path $l(\mathbf{r})$ averaged over a size of order ξ_0. In the region of parameters where *d*-wave superconductivity is sufficiently suppressed by disorder, the spatial dependence of the order parameter can be visualized as a system of superconducting puddles with anomalously large values of the order parameter, which are connected by Joshepson links through non-superconducting metal. The superconductivity inside the puddles may be enhanced because either the electron interaction constant, or the mean free path in the puddles (or both) may be larger than their average values.

Let us assume that the distance between the puddles is larger than both their size and the mean free path. In this case the system is already in a state with the "global *s*-wave" symmetry. Its origin is illustrated qualitatively in Fig.2, where a system of superconduting puddles of arbitrary shape embedded into a metal is shown. The order parameter inside the puddles has *d*-wave symmetry, and the orientation of the gap nodes is assumed to be pinned by the crystalline anisotropy. In a *d*-wave superconductor, in addition to an overall phase of the order parameter, there is an arbitrary sign associated with the internal structure of the pair wave function. Specifically, we adopt a uniform phase convention such that when the phase of the order parameter $\phi_i = 0$, this implies $\Delta(\mathbf{r},\mathbf{r}')$ in puddle i is real and has its positive lobes along the y axis and its negative lobes along the x axis.

The inter-puddle Joshepson coupling originates from the proximity effect in the normal metal. It is characterized by the anomalous Green function $\mathscr{F}(\mathbf{r},\mathbf{r}') \equiv F(\mathbf{r},\mathbf{r}',t=t')$, which is connected to $\Delta(\mathbf{r},\mathbf{r}')$ by the interaction constant. Due to the lack of symmetry at the boundary of a puddle, an *s*-wave component $\mathscr{F}(\mathbf{r}=\mathbf{r}') = \mathscr{F}^{(s)}(\mathbf{r}) \neq 0$ of the anomalous Green function is generated in the neighboring metal. At a distance from the superconductor-normal metal boundary larger than the elastic electron mean free path the anomalous Green function becomes isotropic. In other words, only the *s*-component $\mathscr{F}(\mathbf{r}=\mathbf{r}') = \mathscr{F}^{(s)}(\mathbf{r})$ survives. It is this component that propagates between far separated puddles and determines the Joshepson coupling.

The sign of $\mathscr{F}^{(s)}(\mathbf{r})$ at a normal metal-superconductor boundary, is determined by the sign of the *d*-wave order parameter in the **k**-direction perpendicular to the boundary. Therefore it changes along the boundary of a puddle.

At a distance from an individual i-th puddle larger than its size and smaller than the distance between the puddles the quantity $\mathscr{F}^s(\mathbf{r})$ has a sign $\eta_i = \pm 1$, which depends on the shape of the i-th puddle. This point is illustrated in Fig.2a, where the sign of the anomalous Green function is positive in hatch-marked areas, and negative outside of these areas.

If the distance between puddles is larger than their size, the sign of the Joshepson coupling energy E_{Jos} is determined by a product $\eta_i\eta_j$,

$$E_{Jos} = \sum_{i\neq j} \eta_i\eta_j J_{ij}^{(s)} \cos(\phi_i - \phi_j). \qquad (1)$$

Here indexes i, j label puddles, $J_{ij}^{(S)} > 0$. Eq. 1 represents the Mattis model, which is well known in the theory of spin glasses [11]. The ground state of this model corresponds to

$$\cos(\phi_i) = -\eta_i. \qquad (2)$$

Thus the distribution of $\cos(\phi_i)$ between puddles looks completely random as it is shown in Fig. 2a. However the system is not a glass because it's ground state has a hidden symmetry. In other words if the distances between puddles are bigger than the characteristic size of the puddles, R, the Josephson coupling between puddles inevitably favors globally *s*-wave superconductivity, even though the order parameter on each puddle looks locally *d*-wave -like. It is obvious that at a high concentration of puddles, the order parameter in the ground state has global *d*-wave symmetry (See Fig. 2b.).

At intermediate distances, the situation is more complicated. Areas with different signs of $\mathscr{F}^{(s)}(\mathbf{r})$ mix in a random fashion. We argue that the most important aspects of this complex situation can be modelled by adding to the right hand side of Eq. 1 a term

$$\sum_{i\neq j} J_{ij}^{(d)} \cos(\phi_i - \phi_j), \qquad (3)$$

where $J_{ij}^{(d)} > 0$ characterizes the strength of the exchange interaction between the *d*-wave components of the order parameter. Typically, at small $|\mathbf{r}_i - \mathbf{r}_j|$, $J_{ij}^{(d)} > J_{ij}^{(s)}$, but at large $|\mathbf{r}_i - \mathbf{r}_j|$ the coupling strength $J_{ij}^{(s)}$ decays more slowly than $J_{ij}^{(d)}$. Here $\mathbf{r}_i$ are coordinates of the puddles. Thus it is likely that in this intermediate region the system may exhibit spin glass features and/or coexistence of *d*-wave and *s*-wave ordering. In this article, however, we will not further explore this fascinating but complex aspect of this problem.

To quantify the picture presented above one has to compute the Josephson coupling between a pair of far separated puddles. Since the time that it takes for electrons to travel between puddles is shorter than the characteristic time of fluctuations of the order parameter on individual puddles, one can calculate $J_{ij}^{(s)}$ using the mean-field Usadel equation for the configuration-averaged anomalous Green function $\langle F_\varepsilon^{(s)}(\mathbf{r})\rangle \equiv -i\sin\theta(\varepsilon,\mathbf{r})$ in the metal,

$$\frac{D_{tr}}{2}\partial_{\mathbf{r}}^2\theta(\varepsilon,\mathbf{r}) + i\varepsilon\sin\theta(\varepsilon,\mathbf{r}) = 0. \qquad (4)$$

Here D_{tr} is the transport diffusion coefficient of electrons in the metal, $\mathscr{F}_{\varepsilon}^{(s)}(\mathbf{r})$ is the Fourier transform of $F^{(s)}[\mathbf{r},(t-t'))]$, $\Delta^{(s)}=\lambda^{(s)}\mathscr{F}^{(s)}(\mathbf{r})$, and the brackets $\langle\ldots\rangle$ indicate averaging over random scattering potential between the puddles at a given shape of the puddles. The only, but crucial difference with the conventional case of s-n junctions (See, for example, [13, 14]), is the boundary conditions for Eq. 4 at the normal-superconductor surface, which determine the sign of η_i.

For the case when the size of the puddle is larger than the coherence length and the Andreev reflection on the puddles is effective the boundary conditions for Eq.4 on the d-n boundary have been derived in Ref. [12]. Since the relevant energy for computing the Josephson coupling, $\varepsilon \approx D_{tr}/|\mathbf{r}_i-\mathbf{r}_j|^2$, is much smaller than the value of the order parameter in the puddles, the boundary condition for $\theta(\mathbf{r},\varepsilon)$ is independent of ε and depends only on the angle between the unit vector parallel to the direction of a gap node $\hat{\mathbf{n}}_\Delta$ and a unit vector, $\hat{\mathbf{n}}(\mathbf{r})$, normal to the boundary at point $\mathbf{r}$ at the surface, : $\theta_s(\varepsilon,\mathbf{r})=f[\alpha(\mathbf{r})],\quad \sin[\alpha(\mathbf{r})]\equiv\hat{\mathbf{n}}(\mathbf{r})\cdot\hat{\mathbf{n}}_\Delta$. Here $f(\alpha)$ is a smooth, approximately odd and periodic function, $f(\alpha)\approx -f(-\alpha)$, $f(\alpha)\approx f(\alpha+\pi)$, which grows from $f(\alpha)\approx 0$ at $\alpha=0$, to $f(\alpha)\approx\pm\zeta$ for $\alpha=\pi/4$, where $\zeta\sim 1$. Solving Eq. 4 with these boundary conditions, and using the standard procedure of calculation of the Joshepson energy we get

$$J_{ij}^{s}\sim C\frac{V}{|\mathbf{r}_i-\mathbf{r}_j|^D}\exp(-\frac{|\mathbf{r}_i-\mathbf{r}_j|}{L_T})$$
$$\eta_i=\mathrm{sign}\left\{\int_i ds f(\alpha)\right\} \qquad (5)$$

and $C\sim G_{eff}\frac{D_{tr}}{R^2}$, V is the puddle volume, the integral is taken over the surface of the i_{th} puddle, and G_{eff} is the conductance of a metal of a size of order of the size of the superconducting puddle. In this case the magnitude of the *s*-component of the order parameter generated at the superconductor-normal metal boundary is of order of the magnitude of the *d*-wave component. Thus it is not surprising that the value of $J_{ij}^{(s)}$ in Eq. 1 turns out to be of the same order as in the case of SNS junction.

If the distribution function of the mean-free paths is unbounded, and with certain probabilities one can find arbitrary large values of $l(\mathbf{r})$, the mean field superconducting solution always exists. However, if the puddle concentration is small enough, the transition from the state with global *s*-wave symmetry to the normal metal is triggered by a competition between the inter-puddle Joshepson coupling energy and the thermal (or quantum) fluctuations. Thermal fluctuations destroy the coherence between two puddles when $J_{ij}\sim kT$, which gives us an expression for the critical temperature T_c of the *s*-wave superconductor-metal transition

$$T_c\sim\frac{CV}{R^D}, \qquad (6)$$

where R is the inter-puddle distance.

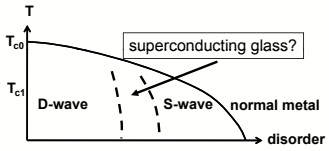

FIGURE 1. Schematic phase diagram for the case when *d*-wave superconducting state is destroyed as a function of increasing disorder strength.

We would like to stress that the existence of the *s*-wave superconducting phase is a generic property of the system because the long-range nature of the decay of Eq. 5 ensures that near the superconductor-normal metal transition and at small enough temperatures the superconducting puddles are separated by a distance larger than their size.

In principle, the situation described above can be realized when grains of *d*-wave superconductors are embedded into a normal metal artificially. In random systems the critical point can be identified by finding the set of "optimal puddles" which lie on the critical links of "the percolating cluster". In this case the properties of the *s*-wave phase and the dependence of the critical temperature T_c on the parameters of the system depends on details of the distribution function of the disordered potential. To illustrate the situation we consider here a simple model where the mean free part $l(\mathbf{r})$ is a random function of coordinates with a Gaussian distribution characterized by an average $\bar{l}$, a variance σl_0, and a correlation length which is of order ξ_0. To be concrete, we consider the 2D case. Then the distance between the puddles becomes of order of their size, the amplitude of fluctuations of the order parameter becomes of order of the average, and the system has a transition to the *s*-wave state when $l\sim l_{c1}$ and $T<T_{c1}$

$$\bar{l_{c1}}-l_0\sim\sigma^2 l,\quad T_{c1}\sim\sigma T_{c0}. \qquad (7)$$

Here T_{c0} is the critical temperature of a pure *d*-wave superconductor. If $l_0-\bar{l}\gg l\sigma^2$ the distance between "the optimal puddles" is much bigger than their size. We can characterize such puddles by a value of the mean free path $l_{opt}>l_0$ averaged over the volume of the puddle. In this case $\Delta_{opt}\sim\Delta_0 l_0/(l_{opt}-l_0)^{1/2}\ll\Delta_0$, the size of the puddle is of order of the zero temperature coherence length $\xi_{opt}\sim\xi_0 l/(l_{opt}-l_0)\gg\xi_0$, and the characteristic distance between the puddles is of order of

$\xi_{opt}\exp[(l_{opt}-\hat{l})^2/2\sigma^2\bar{l}(l_{opt}-l_0)]$. Here Δ_0 is the magnitude of the order parameter in a pure d-wave superconductor at $T=0$. This expression has a minimum at $(l_{opt}-l_0)\sim(l_0-\bar{l})$, and therefore $R_{opt}\sim\exp[(l_0-\bar{l})/l_0\sigma^2]$. Using Eq. 6 we get

$$T_c\sim T_{c0}\sigma\exp[-\frac{(l_0-\bar{l})}{l_0\sigma^2}]. \tag{8}$$

At very small values of T_c the phase transition between the s-wave superconducting phase and the normal metal is triggered by quantum fluctuations of the order parameter. We now outline the procedure for determination of the location of the quantum critical point under these circumstances. Quantum fluctuations necessarily destroy the superconducting order in an isolated puddle. Thus, although the superconducting susceptibility of an individual puddle, χ_i, can, under some circumstances, be large, the transition to the globally phase coherent superconducting state is ultimately triggered by the Josephson coupling between puddles. Let us introduce a dimensionless coupling between two puddles, i and j,

$$X_{i,j}\equiv\chi_i J_{i,j}\chi_j J_{j,i}. \tag{9}$$

Two puddles fluctuate essentially independently of each other if $|X_{i,j}|\ll 1$, and they are phase locked to each other if $|X_{i,j}|\gg 1$. The transition to a globally phase coherent state occurs as a function of $\bar{l}$ at the critical value, $\bar{l}=l_{c1}(T=0)$, at which an infinite cluster of puddles is coupled together by links with $X_{i,j}\sim 1$. For an ordered array of puddles, the quantum superconductor-metal transition was discussed in this light in [16, 17].

The expression for the susceptibility of a puddle depends on its radius. It is possible to show that the characteristic size of the optimal puddles $\xi_{opt}\sim\xi_0 l/(l_{opt}-l_0)$ is of order of the superconducting coherence length which corresponds to the value order parameter in the puddle. In this case the susceptibility of superconducting puddles was investigated in many papers (See for example [19, 20, 21, 22, 23, 24, 16]). To be concrete we consider 2D case where [16]

$$\ln\chi/\Delta_0\sim\sqrt{G} \tag{10}$$

where $G\gg 1$ is the conductance of the film per square.

Thus in a generic situation in the neighborhood of the transition, rare puddles with exponentially large susceptibilities play a special role. Specifically, the optimal puddles are those in which l lies in an interval, $l_{opt}-\Delta l_{opt}<l<l_{opt}+\Delta l_{opt}$. Here both the optimal value, l_{opt}, and the width of the interval, $\Delta\, l_{opt}$, are determined by maximizing the quantity $X_{opt}=\chi^2_{opt}J^2_{opt}$ with respect to these parameters, where χ_{opt} is the susceptibility of a puddle with $l=l_{opt}$, and J_{opt} is the typical value of the Josephson coupling between two nearest-neighbor optimal puddles. Finally we find the critical value of $l_{c1}(T=0)$ from the requirement that, after maximizing, $max\{X_{opt}\}\approx 1$.

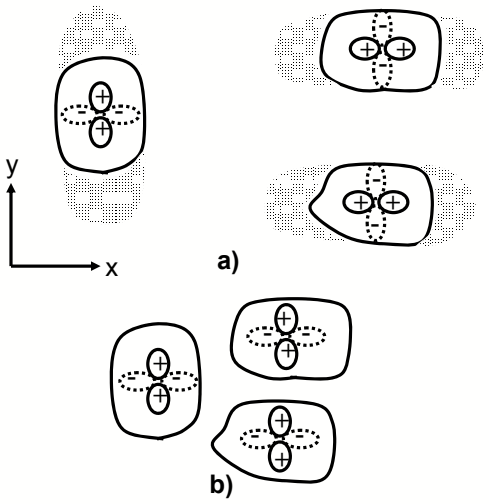

FIGURE 2. A qualitative illustration of the global d-wave to s-wave transition. Solid lines represent boundaries of d-wave superconducting puddles embedded into a normal metal. Hatch marked areas indicate the areas were the s-wave component of the anomalous Green function $F^s(\mathbf{r},\mathbf{r})$ is positive. Outside these areas $F^s(\mathbf{r},\mathbf{r})$ is negative. a) The case of small puddle concentration when the system has s-wave global symmetry. b) The case of big puddle concentration when the system has a global d-wave symmetry.

$$\frac{l_0-l_{c1}(T=0)}{l_0}\approx G\sigma^2<1 \tag{11}$$

Thus, a generic feature of the s-n transition is that it takes place when the distance between optimal puddles is exponentially large [18].

The cuprate high-temperature superconductors are the best established example of a d-wave superconductor. Here, the critical temperature, T_c, is known to vary strongly as a function of the doped hole concentration, x, producing two quantum critical points at which T_c vanishes: a lower critical doping concentration, x_1, on the "underdoped" side, and an upper critical concentration, x_2, on the "overdoped" side of the phase diagram. On the underdoped side of the superconducting dome, with increasing underdoping, these materials frequently appear to undergo a superconductor to insulator transition [7, 6, 8]. Thus, the present considerations are not applicable. We assume, some of the more robust of our findings apply to the cuprates as $T_c\to 0$ with overdoping. There are a number of interesting predictions we can make. 1) There should be a transition from a globally d-wave to a globally s-wave superconducting state at a doping concentration $x=x_2$. (Some evidence of such a transition may already be present in the experiments of Ref. [9].) 2) In the metallic state with $x>x_2$, the conductivity at low temperature should diverge as $x\to x_2$, the Hall resis-

tance should vanish, and the Weideman-Franz law should be increasingly strongly violated.

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