The problem of Macroscopic Charge Quantization in the Coulomb Blockade

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Abstract. Based on the Ambegaokar-Eckern-Schön approach to the Coulomb blockade, we develop a complete quantum theory of the single electron transistor. We identify a previously unrecognized physical observable in the problem that, unlike the usual average charge on the island, is robustly quantized for any *finite* value of the tunneling conductance as the temperature goes to absolute zero. This novel quantity is fundamentally related to the non-symmetrized current noise of the system. Our results display all of the superuniversal topological features of the θ angle concept that previously arose in the theory of the quantum Hall effect.

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The Coulomb blockade has traditionally been viewed as an experimental demonstration of "macroscopic charge quantization" [1]. The standard experimental set-up is the single electron transistor (SET) [2] which is a mesoscopic metallic island coupled to a gate and connected to two metallic reservoirs by means of tunnelling contacts with a total conductance g (see Fig. 1).

Even though in the absence of tunneling the problem is generally well understood, it has nevertheless been shown that for any finite value of g, no matter how small, the averaged charge Q on the island is *un*-quantized as the temperature (T) goes to absolute zero [3]. This raises fundamental questions about the exact significance of the experiments and the physical quantities in which the Coulomb blockade is usually expressed.

In a recent investigation [4] the authors reported complete quantum theory of the SET. This theory is motivated by the formal analogies that exist between the Ambegaokar-Eckern-Schön (AES) theory on the one hand, and the theory of the quantum Hall effect [5] on the other. By studying the sensitivity of the system to changes in the boundary conditions it was shown that the AES theory generally defines two physical observables, the SET conductance g' and a novel quantity termed q' that is fundamentally related to the current noise in the SET. The q' is in all respects same as the Hall conductance in the quantum Hall effect and, unlike the averaged charge Q on the island, it is robustly quantized in the limit $T \to 0$ independent of the value of g.

The physical parameters g' and q' set the stage for a unifying renormalization theory of the SET within which the various disconnected pieces of existing computational knowledge of the AES theory can in general be understood. In this paper we review some of the main ingredients of this theory and provide some more detail on the development of the scaling diagram in the g'- q' plane.

AES MODEL

The action involves a single abelian phase $\phi(\tau)$ describing the potential fluctuations on the island $V(\tau) = i\dot{\phi}(\tau)$ with τ denoting the imaginary time [6]. The theory is defined by

$$Z = \int \mathscr{D}[\phi] e^{-S[\phi]}, \quad S[\phi] = S_d + S_t + S_c. \qquad (1)$$

The action S_d describes the tunneling between the island and the reservoirs

$$S_d[\phi] = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2\, \alpha(\tau_{12}) e^{-i[\phi(\tau_1)-\phi(\tau_2)]}. \qquad (2)$$

Here, $\beta = 1/T$, $\tau_{12} = \tau_1 - \tau_2$ and $g = g_l + g_r$, where $g_{l,r}$ denotes the dimensionless bare tunneling conductance between the island and left/right reservoir (see Fig. 1). The kernel $\alpha(\tau)$ is usually expressed as $\alpha(\tau) = (T/\pi)\sum_n |\omega_n| e^{-i\omega_n \tau}$ with $\omega_n = 2\pi T n$. The part S_t describes the coupling between the island and the gate and S_c is the effect of the Coulomb interaction between the electrons

$$S_t[\phi] = -2\pi i q W[\phi], \quad S_c[\phi] = \frac{1}{4E_c}\int_0^\beta d\tau\, \dot{\phi}^2. \qquad (3)$$

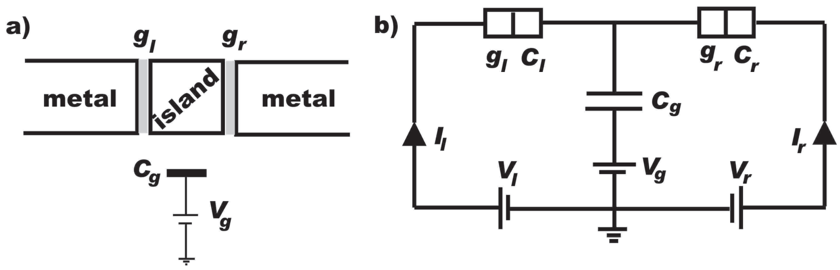

FIGURE 1. a) Sketch of the SET device. b) Equivalent circuit of the SET.

Here, q is the external charge and $W[\phi] = 1/(2\pi)\int_0^\beta d\tau\dot{\phi}$ is the winding number or *topological charge* of the ϕ field. For the system in equilibrium the winding number is strictly an integer [7] which means that Eq. (3) is only sensitive to the *fractional* part $k - 1/2 < q \leqslant k + 1/2$ of the external charge q.

Eq. (2) has classical finite action solutions $\phi_W(\tau)$ with a non-zero winding number that are completely analogous to Yang-Mills instantons. The general expression for winding number W is given by [8, 9]

$$e^{i\phi_W(\tau)} = e^{-i2\pi T\tau} \sum_{a=1}^{|W|} \frac{e^{i2\pi T\tau} - z_a}{e^{-i2\pi T\tau} - z_a^*}. \tag{4}$$

For instantons ($W > 0$) the complex parameters z_a are all inside the unit circle and for anti-instantons ($W < 0$) they are outside. The classical action is finite $S_d[\phi_W] + S_t[\phi_W] = g|W|/2 - 2\pi qWi$ leaving the set of parameters $\{z_1, \ldots, z_{|W|}\}$ as $2|W|$ zero modes in the problem.

On the weak coupling side (large g) the dominant contribution comes from widely separated single instantons with $W = \pm 1$. This leads to the dilute instanton gas approximation where $\arg z/2\pi T$ is recognized as the *position* of the single instanton and $\lambda = (1 - |z|^2)\beta$ equals the *scale size* or the duration of the potential pulse $i\dot{\phi}_W(\tau)$. The main effect of S_c in Eq. (3) is to provide a cut-off for large frequencies or small scale sizes λ.

KUBO FORMULAE FOR THE OBSERVABLE PARAMETERS

To develop a general quantum theory of the SET that encompasses both the weak and strong coupling aspects of the AES model we make use of the fact that $\tilde{\phi}(\tau) = \omega_n\tau$ satisfies the classical equation of motion of Eq. (1). By employing $\tilde{\phi}(\tau)$ as a background field then the effective action $S'[\tilde{\phi}]$ is properly defined in terms of a series expansion in powers of ω_n. Retaining only the lowest order terms in the series we can write

$$S'[\tilde{\phi}] = \beta\left[\frac{g'}{4\pi}|\omega_n| - iq'\omega_n + O(\omega_n^2)\right]. \tag{5}$$

The quantities of physical interest are the parameters g' and q' with $k - 1/2 < q' \leqslant k + 1/2$. They are formally given in terms of the linear response expressions according to [4, 10]

$$g' = 4\pi\,\mathrm{Im}\,\frac{\partial K^R(\omega)}{\partial\omega}\bigg|_{\omega=0}, \qquad q' = Q + \mathrm{Re}\,\frac{\partial K^R(\omega)}{\partial\omega}\bigg|_{\omega=0}. \tag{6}$$

Here, $Q = q + i\langle\dot{\phi}\rangle/(2E_c)$ denotes the average charge on the island. The function $K^R(\omega)$ is defined as the analytic continuation $i\omega_n \to \omega + i0^+$ of the following expression

$$K(i\omega_n) = -\frac{g}{4\beta}\int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n\tau_{12}}\alpha(\tau_{12})D(\tau_{21}) \tag{7}$$

with $D(\tau_{12}) = \langle\exp i[\phi(\tau_2) - \phi(\tau_1)]\rangle$. The function $K^R(\omega)$ is written more explicitly in terms of the retarded propagator $D^R(E)$ [10]

$$K^R(\omega) = g\int\frac{dEdE'}{4\pi^3}E'\frac{n_B(E') - n_B(E)}{E - E' + \omega + i0^+}\,\mathrm{Im}\,D^R(E) \tag{8}$$

where $n_B(E) = [\exp(\beta E) - 1]^{-1}$ is the Bose-Einstein distribution.

The quantities g' and q' probe the low energy dynamics of the SET since they are, by construction, a measure for the sensitivity of the system to infinitesimal changes in the boundary conditions. Eq. (6) defines in fact exactly the same quantities that one normally would obtain in ordinary linear response theory [4, 10]. For example, g' is same as the Kubo formula [11, 12, 13] relating a small potential difference V between the reservoirs to the current $\langle I\rangle$ across the island: $\langle I\rangle = e^2GV/h$ where the SET conductance $G = g_lg_rg'/(g_l + g_r)^2$ and h is Planck's constant. The new quantity q' is more transparently written in terms of the quantum current noise according to

$$q' = Q - \frac{(g_l + g_r)^2}{2g_lg_r}i\frac{\partial}{\partial V}\int_{-\infty}^0 dt\langle[I(0), I(t)]\rangle \tag{9}$$

in the limit $V \to 0$.

WEAK COUPLING REGIME, $g' \gg 1$

By evaluating Eq. (6) in a series expansion in powers of $1/g$ one obtains the well-known perturbative results for $g'(T)$ [14, 15, 16]. The new quantity q' is unaffected by the quantum fluctuations. To establish the renormalization of q' it is necessary to include the effect of instantons. Extending the methodology of Ref. [5] we find [10]

$$\begin{aligned} \mathrm{Im} D^R(\omega) &= \pi\beta\omega\delta(\omega)\left[1-\frac{2}{g}\ln\frac{gE_c e^{\gamma}}{2\pi^2 T}\right] \\ &+ \mathrm{Im}\left(\frac{2\pi i/g}{\omega+i0^+}-\frac{2\pi i/g}{\omega+igE_c/\pi}\right) \\ &- \frac{g^2E_c}{\pi^2 T}e^{-g/2}\,\mathrm{Re}\left(\left[\pi\beta\omega\delta(\omega)\right.\right. \\ &- \left.\left.\frac{1}{\omega+i0^+}+\frac{1}{\omega+i2\pi T}\right]e^{-i2\pi q}\right) \end{aligned} \quad (10)$$

where $\gamma = 0.577\ldots$ denotes the Euler constant. The first two lines in Eq. (10) are the one-loop perturbative results whereas the terms proportional to $\exp(-g/2)]$ are typically instanton terms with $W=\pm 1$. Using Eq. (10) we obtain Eq. (8) as follows

$$\begin{aligned} K^R(\omega) &= \frac{i\omega g}{4\pi}\left[1-\frac{2}{g}\ln\frac{egE_c}{2\pi^2 T}+\frac{2}{g}\psi\left(1-\frac{i\omega}{2\pi T}\right)\right] \\ &- \frac{g^3E_c}{2\pi^2}e^{-g/2}e^{i2\pi q}\left[\psi(1)-\psi\left(1-\frac{i\omega}{2\pi T}\right)\right] \\ &- \frac{g^3E_c}{2\pi^2}e^{-g/2}\cos 2\pi q\sum_{n>1}^{n_{max}}\frac{1}{n} \\ &- \sum_{n>0}^{n_{max}}\frac{gE_cT}{2\pi^2 Tn+gE_c} \end{aligned} \quad (11)$$

where $\psi(z)$ denotes the Euler di-gamma function and $n_{max} \approx gE_c/T$. The final results for the observable parameters of Eq. (6) can be written as follows

$$g'(T) = g-2\ln\frac{gE_ce^{\gamma+1}}{2\pi^2 T}-\frac{g^3E_c}{6T}e^{-g/2}\cos 2\pi q \quad (12)$$

$$q'(T) = q-\frac{g^3E_c}{24\pi T}e^{-g/2}\sin 2\pi q. \quad (13)$$

These results describe the Coulomb blockade in the weak coupling regime $g' \gg 1$ or at high temperatures $T \gg g^3E_ce^{-g/2}$ such that the amplitude of oscillations with varying external charge q is small. These oscillations are nevertheless much stronger than those appearing in the average charge Q on the island [17, 18, 16]

$$Q(T) = q-\frac{g^2}{\pi}e^{-g/2}\ln\left(\frac{E_c}{2\pi^2 e^{\gamma}T}\right)\sin 2\pi q. \quad (14)$$

Eqs (12) and (13) are completely analogous to the instanton results originally obtained in the theory of the quantum Hall effect [19, 20] and recently studied experimentally [21]. It should be mentioned that the result for the SET conductance Eq. (12) coincides with that obtained in Ref. [13].

STRONG COUPLING REGIME, $g' \ll 1$

Near the point $g=0$ and $q=k+1/2$ one can project the theory onto the low energy states with $Q=k$ and $Q=k+1$ respectively [3]. The AES theory is then mostly elegantly described by the spin 1/2 effective action [10]

$$\begin{aligned} S_{pf} &= \beta E_c q^2+\beta\frac{\Delta}{2}+\int_0^{\beta}d\tau\overline{\psi}\left(\partial_\tau-\eta+\frac{\Delta}{2}\sigma_z\right)\psi \\ &+ \frac{g}{4}\int_0^{\beta}d\tau_1 d\tau_2\alpha(\tau_{12})[\overline{\psi}(\tau_1)\sigma_-\psi(\tau_1)] \\ &\times[\overline{\psi}(\tau_2)\sigma_+\psi(\tau_2)]. \end{aligned} \quad (15)$$

Here, ψ and $\overline{\psi}$ are the Abrikosov's two-component pseudofermion fields [22, 23]. The quantity $\Delta = E_c(2k+1-2q) > 0$ is the energy gap between the charging levels with $Q=k$ and $Q=k+1$ respectively and σ_j with $j=x,y,z$ stands for the Pauli matrices and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$.

We have introduced a chemical potential η in Eq. (15) in order to eliminate the contributions from the non-physical states with pseudofermion number $N_{pf} \neq 1$. This is accomplished by taking the limit $\eta \to -\infty$ at the end of all computations.

Comparing Eqs. (1) and (15) we identify the spin operators $\overline{\psi}(\tau)\sigma_\pm\psi(\tau)$ with the AES operators $\exp(\pm i\phi(\tau))$ projected onto the states with $Q=k$ and $Q=k+1$ respectively. It is interesting to notice that the effective action (15) is similar to the XY case of the Bose-Kondo model for spin $s=1/2$ [24, 25]. Similar to the one loop renormalization group procedure described in Ref. [25] we sum the leading logarithms and find

$$\begin{aligned} \mathrm{Im} D^R(\omega) &= \frac{\pi}{\gamma^2}\delta(\omega-\Delta')\tanh\frac{\beta\Delta'}{2}, \\ \gamma^2 &= 1+\frac{g}{2\pi^2}\ln\frac{\Lambda}{\max\{\Delta',T\}}. \end{aligned} \quad (16)$$

Here, $\Delta' = \Delta/\gamma^2$ denotes the renormalized energy gap and Λ is a high energy cut-off of order E_c. We obtain the following result for Eq. (8)

$$\begin{aligned} K^R(\omega) &= \frac{g}{\gamma^2}\frac{\omega+\Delta'}{4\pi^2}\tanh\frac{\beta\Delta'}{2}\left[\psi\left(1-i\frac{\omega+\Delta'}{2\pi T}\right)\right. \\ &- \left.\psi\left(1-\frac{i\Delta'}{2\pi T}\right)-\frac{2\pi}{\Delta'}Y(\Delta')\right] \end{aligned} \quad (17)$$

with $Y(\Delta') = T\sum_{\omega_n>0}\omega_n\Delta'/(\omega_n^2+\Delta'^2)$. Based on Eq. (17) we obtain the following expressions for the physical observables in Eq. (6)

$$g'(T) = \frac{g}{2\gamma^2}\frac{\beta\Delta'}{\sinh\beta\Delta'}, \qquad (18)$$

$$q'(T) = Q(T) - \frac{\gamma^2-1}{2\gamma^2}\tanh\frac{\beta\Delta'}{2}. \qquad (19)$$

Here, Eq. (18) and $Q(T)$ are the same as the results obtained in Ref. [26]

$$Q(T) = \frac{1}{2}\left(1-\frac{1}{\gamma^2}\tanh\frac{\beta\Delta'}{2}\right). \qquad (20)$$

In the limit $T=0$ we find

$$Q(T=0) = \frac{g}{4\pi^2}\ln\frac{\Lambda}{\Delta'}\Big/\left(1+\frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}\right) \qquad (21)$$

which is the result obtained by Matveev [3]. It says that in the presence of tunneling $g\neq 0$ the average charge $Q(T)$ on the island is *un-* quantized. On the other hand, Eqs. (19) and (20) imply

$$q'(T) = k + \frac{1}{1+e^{\beta\Delta'}} \qquad (22)$$

indicating that the novel physical quantity q' is *robustly quantized* as one moves away from quantum criticality.

Finally, at lower temperatures $T\lesssim\Delta'$ the *inelastic co-tunneling* processes become important [27]. The lowest order correction to $D^R(\omega)$ due to inelastic cotunneling is obtained as follows [10]

$$\delta D^R_{\rm inel}(\omega) = \frac{g}{4\pi\gamma^4}\frac{i\omega}{(\omega-\Delta'+i0^+)^2}. \qquad (23)$$

Based on this result we compute the corrections to the physical observables using Eq. (8) and Eq. (6) and the result is

$$\delta g'_{\rm inel}(T) = \frac{g^2}{4\pi^2\gamma^4}\frac{\partial}{\partial\Delta'}\left[\frac{\beta\Delta'^2}{2\pi}\psi'\left(1+\frac{i\beta\Delta'}{2\pi}\right)-\Delta'\right]$$

$$\delta q'_{\rm inel}(T) = \frac{g^2}{4\pi\gamma^4}\Delta'\frac{\partial^2}{\partial\Delta'^2}\left(\Delta'\coth\frac{\beta\Delta'}{2}\right). \qquad (24)$$

In the regime $T\lesssim\Delta'$ we find

$$\delta g'_{\rm inel}(T) = \frac{g^2}{48\gamma^4}\frac{T^2}{\Delta'^2} \qquad (25)$$

$$\delta q'_{\rm inel}(T) = \frac{g^2}{2\pi\gamma^4}\frac{\Delta'^2}{T^2}e^{-\beta\Delta'}. \qquad (26)$$

Eq. (25) coincides with the result found in Ref. [26]. As a final remark we emphasize that the results for cotunneling cannot be extrapolated all the way down to the regime of exponential localization $T\ll\Delta'$ which is clearly beyond the range of validity of the strong coupling expansion.

SUMMARY AND CONCLUSIONS

The temperature dependence of the physical observables g' and q' can in general be expressed in terms of renormalization group β functions

$$\frac{dg'}{d\ln\beta} = \beta_g(g',q'), \quad \frac{dq'}{d\ln\beta} = \beta_q(g',q'). \qquad (27)$$

From Eqs. (12), (13) we extract for the weak coupling regime

$$\beta_g = -2 - \frac{\pi^2 e^{-\gamma-1}}{3}g'^2 e^{-g'/2}\cos 2\pi q', \qquad (28)$$

$$\beta_q = \frac{\pi e^{-\gamma-1}}{12}g'^2 e^{-g'/2}\sin 2\pi q'. \qquad (29)$$

From Eqs. (18) and (22) we obtain the following strong coupling behavior near $q'=k+1/2$

$$\beta_g = -\frac{g'^2}{\pi^2}, \qquad \beta_q = \left(q'-k-\frac{1}{2}\right)\left(1-\frac{g'}{\pi^2}\right). \qquad (30)$$

This result shows that $q'=k+1/2$ and $g'=0$ is the *critical* fixed point of the AES theory with g' a marginally irrelevant scaling variable.

Eqs (29) and (30) together determine a unifying scaling diagram of the SET in the g' and q' plane as illustrated in Fig. 2. The stable strong coupling fixed points $g'=0$ and $q'=k$ clearly indicate that the AES theory generally displays the Coulomb blockade as T goes to absolute zero. This scaling phenomenon is fundamentally different from semiclassical picture of the Coulomb blockade since it elucidates the discrete nature of the electronic charge which is independent of tunneling.

In conclusion, the AES model is an extremely interesting and exactly solvable example of a θ angle concept that displays all the *super universal* topological features that have arisen before in the context of the quantum Hall liquids [5, 28] as well as quantum spin liquids [29]. These include not only the existence of *gapless* or critical excitations at $q'=k+1/2$ (or $\theta=\pi$) but also the *robust* topological quantum numbers that explain the "macroscopic charge quantization" of the SET at zero temperature and finite values of g.

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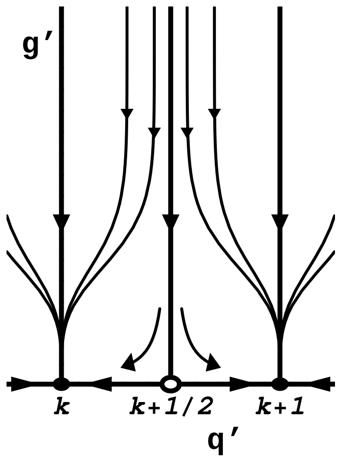

FIGURE 2. Unified scaling diagram of the Coulomb blockade in terms of the SET conductance g' and the q'. The arrows indicate the scaling toward $T = 0$ (see text).

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