Statistics of Measurement of Non-commuting Quantum Variables

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Abstract. We address continuous weak linear quantum measurement and argue that it is best understood in terms of statistics of the outcomes of the linear detectors measuring a quantum system, for example, a qubit. We develop a proper formalism to evaluate the statistics of such measurement. Generally, we are able to evaluate the joint probability distribution of the detector outcomes and the qubit variables. We concentrate on two setups. The application of our method to the setup where a single pseudospin component is measured gives a comphrehensive picture of quantum non-demolition measurement. More interesting setup consists of a qubit and three independent detectors that simultaneously monitor three non-commuting operator variables, those corresponding to three pseudo-spin components of the qubit. When analyzing the distribution in the limit of big values of the outcomes, we reveal a high degree of correspondence between the three outcomes and three components of the qubit pseudo-spin after the measurement. This enables a high-fidelity monitoring of all three components. We discuss the relation between the monitoring described and the algorithms of quantum information theory that use the results of the partial measurement. The formalism is based on Feynman-Vernon approach, roots in the theory of full counting statistics, and boils down to a Bloch-Redfield equation augmented with counting fields.

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INTRODUCTION

The theory of quantum measurement, being a foundation of quantum physics, is attracting more and more attention [1]. Intrinsic paradoxes [2] are definitely a main reason for studying quantum measurements. More motivation comes from the practical needs to understand the real solid-state based devices [3, 4] developed for quantum computing [5]. Measurements in solid-state setups may provide access to extra variables that facilitate the read out of the quantum information stored in the elementary two-level quantum systems (qubits). The concept of continuous weak linear measurement (CWLM), where the interaction between the detector and the measured system is explicit and sufficiently weak, has been recently elaborated in context of the solid state quantum computing [6, 7, 8, 9, 10]. CWLM provides a universal description of the measurement process and is based on general linear response theory [11]. It applies to a large class of linear detectors: From common amplifiers to more exotic on-chip detectors such as quantum point contact [12], superconducting SET transistors [13], generic mesoscopic conductors [14], fluxons in a Josephson transmission line to measure a flux qubit [15, 16].

It is an important feature of CWLM that the (quantum) information is transferred from a quantum system being measured — a qubit — to other degrees of freedom: those of the detector. The outcome of the measurement is thus represented by the detector degrees of freedom rather than those of the qubit. We will address both the statistics of the outcomes and joint statistics of the outcomes and the qubit degrees of freedom.

We stress the difference between the detector outcomes and the outcomes of a projective measurement of a qubit. In distinction from the result of a projective measurement, the detector outcome is not discrete, since the detector output (for instance, voltage or current) is a continuous variable. The outcomes do not even have to correlate with the state of the qubit if the detector is uncoupled. Further, the detector variables are subject to noise not related to the qubit. Owing to the feedback of the detector at the qubit, this noise affects the qubit too.

In comparison with the text-book projective measurement that instantly provides a result and projects the system onto the state corresponding to the result, the CWLM takes time both to accumulate the information and to distort the qubit. The time τ_m required to obtain a sufficiently accurate measurement result is called "measurement time" and is a characteristic of a CWLM setup. It is not a duration of an individual measurement in this setup: the latter may vary. The distortion is due to the inevitable back action of the detector and is characterized by the dephasing rate Γ_d. It has been shown [6, 7, 8] that for an optimized — quantum limited — detector $\tau_m\Gamma_d = 1/2$ while the "measurement time" τ_m greatly exceeds $1/2\Gamma_d$ for less optimal detectors.

In the context of quantum information theory, CWLM may be understood as an interaction of the qubit with infinitely many ancillary qubits representing the detector degrees of freedom. Each ancilla is brought to weakly interact with the qubit for a short time and is subsequently measured. Owing to the interaction, the quantum state of the ancillae is entangled with the state of the qubit. The detector output is proportional to the sum of the measurement results of a large set of ancillae. This allows to transfer quantum information from the qubit to the detector without formal projective measurement of the qubit. Therefore the peculiarities of the CWLM can be understood in the framework of a projective measurement, although a more complicated one involving the detector degrees of freedom. The CWLM can be thus seen as a build-up of an entanglement between the qubit and the detector. An outcome of an individual CWLM is the detector output accumulated during the time interval of a certain duration τ_d. Any CWLM can be described as a generalized quantum measurement, that involves qubit and detector degrees of freedom.

The outcome randomly varies from measurement to measurement. We argue here that studying statistics of the measurement outcomes of a CWLM is the best way to understand and characterize such a measurement. This is especially important for the simultaneous measurement of non-commuting variables (say, A and B) we concentrate on in this work. In this case, the text-book projective measurement can not help to predict the statistics of the results: it would depend on the order of measurements of A and B. This property of the measurements in non-commuting bases enables most quantum cryptography [17] algorithms and has been extensively elucidated in Ref. [18].

One can straightforwardly realize in experiment a CWLM of a quantum system where A and B are measured simultaneously. If A and B commute, the statistics of the outcomes of sufficiently long CWLM corresponds to the predictions of projective measurement scheme (see Sec. III). The projective measurement scheme loses its predictive power if A and B do not commute. The reason is that the order of measurement of A and B is not determined in the course of a continuous measurement. The statistics of CWLM outputs thus can not be straightforwardly conjectured and has to be evaluated from the quantum mechanical treatment of the whole system consisting of the qubit and the detectors.

In a sharp contrast to the case of commuting variables, the most probable outcome of a sufficiently long CWLM of non-commuting variables does not depend on the qubit state. Therefore it provides no information about the qubit. The information is however hidden in the statistics of random outcomes. Recently, the simultaneous acquisition of two non-communing observables was investigated in the framework of CWLM [9], and the correlation of the random output of two detectors was found to be informative. Not only noise, but the whole full counting statistics (FCS) of the non-commuting measurements has been recently addressed for an example of many spins traversing the detectors [19].

RESULTS IN SHORT

We develop the necessary formalism for the measurement statistics. Our approach stems from the FCS theory of electron transfers [20] in the extended Keldysh formalism [21], which has been recently discussed [22] in the context of the quantum measurement. At first step, we obtain a Feynman-Vernon action to describe the fluctuations of the input and output variables of the detector(s). In the relevant limit, the action is local in time. So at the second step we reduce the path integral to the solution of a differential equation that appears to be a Bloch-Redfield equation augmented with the counting field. We exemplify the formalism addressing a relatively simple case of quantum non-demolition (QND) measurement [23]. We evaluate the distribution of the outcomes for a single detector and understand the statistics of a recently proposed quantum un-demolition measurement[10]. The main results concern the statistics of measurement of non-commuting variables for the case of three independent detectors measuring the three components of the qubit pseudo-spin. We find the statistical correspondence between the three outcomes and three wavefunction components after the measurement. The correspondence is characterized by a fidelity that generally increases with the magnitude of the outcomes reaching the ideal value 1 in the limit of large magnitudes. Since very large outcomes are statistically rare and require long waiting times, this result could be of a purely theoretical value. To prove the opposite, we have evaluated the fidelity at moderate magnitudes of outcomes and measurement durations τ_d and we were able to demonstrate the fidelity of 0.95 for $\tau_d \simeq 7\tau_m$. We term this "quantum monitoring". Ideally, the result of the quantum monitoring is a pure state of the qubit and three numbers (detector outputs) giving the polarization of the state. The same result can be also achieved by preparing the qubit state of the known polarization, for instance, by a projective measurement along a certain axis. The difference is that in the case of preparation the polarization axis is known to the observer in advance, while in the case of monitoring it is not so: both the three numbers and the state emerge from dynamics of the quantum system that encompasses the qubit and the detectors. A fuller account of the results can be found in [28].

METHOD

We start with a single detector setup. The Hamiltonian reads as follows:

$$H = H_q + H_{int} + H_d \,, \tag{1a}$$

$$H_q = \sum_{i=1}^{3} H_i \hat{\sigma}_i; \; H_{int} = \hat{\sigma}_3 \hat{Q} \,. \tag{1b}$$

Here, H_q is the Hamiltonian of the qubit in terms of Pauli matrices $\hat{\sigma}_i$ (i = 1, 2, 3) corresponding to three components of the qubit pseudo-spin. H_{int} gives the interaction between the detector and the third component of the pseudo-spin of the qubit, $\hat{Q}$ being the detector *input* variable. H_d is the Hamiltonian of the detector which we do not specify assuming linear dynamics of the detector variables. Our goal is the statistics of the detector output variable $\hat{V}$. Following [29] we introduce a counting field $\chi(t)$ coupled to the output variable $\hat{V}$ and use Keldysh scheme with different Hamiltonians $H^{\pm}$ on two parts of the Keldysh contour[26], $H^{\pm} = H \pm \hbar\chi(t)\hat{V}/2$. $\pm$ corresponding to the upper/lower branch of the contour. The counting field $\chi(t)$ plays a role of the variable in the probability-generating function of the detector outcomes $V(t)$, this function is given by:

$$Z(\{\chi(t)\}) = \mathrm{Tr}\left(\overrightarrow{\mathrm{T}} e^{\frac{-i}{\hbar}\int dt H^{+}} \hat{\mathrm{R}}(0)\, \overleftarrow{\mathrm{T}} e^{\frac{i}{\hbar}\int dt H^{-}}\right). \tag{2}$$

$\mathrm{Tr}(\cdots)$ implying the trace over both detector and qubit variables. Here, $\overrightarrow{\mathrm{T}}(\overleftarrow{\mathrm{T}})$ denotes time (reversed) ordering in evolution exponents and $\hat{\mathrm{R}}(0)$ is the initial density matrix of the qubit and detector. Next we employ the path integral representation for the probability-generating function [29]. The integral is over the detector variables,

$$\begin{aligned} Z(\{\chi(t)\}) &= \int \mathscr{D}\bar{X}^{+}\mathscr{D}\bar{X}^{-} e^{A_d(\{\bar{X}^{+}(t),\bar{X}^{-}(t),\bar{\chi}(t)\})} \\ &\quad \times \mathrm{Tr}_{\text{qubit}}\left(\overrightarrow{\mathrm{T}} e^{-\frac{i}{\hbar}\int dt\,(H_q+\hat{\sigma}_3\hat{Q}^{+}(t))} \hat{\rho}(0) \right. \\ &\quad \left. \times \overleftarrow{\mathrm{T}} e^{\frac{i}{\hbar}\int dt\,(H_q+\hat{\sigma}_3\hat{Q}^{-}(t))}\right). \end{aligned} \tag{3}$$

Here, $\bar{X}^{\pm}(t)$ are two-dimensional vectors of the detector variables $\bar{X}^{\pm}(t) = (Q^{\pm}(t), V^{\pm}(t))^T$, $\mathscr{D}\bar{X}^{\pm} = \prod_t \mathscr{D}\bar{X}^{\pm}(t)$, $\hat{\rho}(0)$ is the qubit density matrix. The action A_d is bilinear in $\bar{X}^{\pm}$ to describe linear dynamics of the detector. Following common assumptions about CWLM, [7, 8] we assume instant detector responses and white noises to arrive at

$$\begin{aligned} A_d &= \int dt[-\frac{1}{2}\bar{x}^T(t)(\check{a}^{-1})^T \check{S}\check{a}^{-1}\bar{x}(t) \\ &\quad + i\bar{X}^T(t)\check{a}^{-1}\bar{x}(t) + i\bar{\chi}^T(t)\bar{X}(t)], \end{aligned} \tag{4}$$

where we switch to the "quantum" ($\bar{x}$) and "classical"($\bar{X}$) variables defined as follows: $\bar{x} = (\bar{X}^{+} - \bar{X}^{-})/\hbar$, $\bar{X} = (\bar{X}^{+} + \bar{X}^{-})/2$. Here, $\bar{\chi} = (0, \chi)^T$, the 2×2 matrices $\check{a}$, $\check{S}$ give the response functions and noises of the detector respectively. S_{11} is the noise of the input variable responsible for the backaction of the detector and decoherence of the qubit; S_{22} is the output noise and $S_{12} = S_{21}$ presents the correlation of these two noises. a_{12} determines the detector response on the qubit pseudo-spin, $\langle\hat{V}\rangle = a_{12}\langle\hat{\sigma}_3\rangle$. Other response functions a_{21}, a_{22}, a_{11} are respectively related to reverse gain, output and input impedances of the detector and are not of immediate interest for us. The detector is characterized with the dephasing rate $\Gamma_d = 2S_{11}/\hbar^2$ and the "measurement time" $\tau_m = S_{22}/a_{12}^2$.[31, 7] The Cauchy-Schwartz inequality $4S_{11}S_{22} - 4S_{12}^2 \geq \hbar^2 a_{12}^2$ [7, 8] guarantees $\tau_m\Gamma_d \geq 1/2$.

It is important for further advance that the action (4) is local in time. In this case, the path integral in (3) can be taken at each time slice separately. The result of integration is expressed in terms of the solution of a local-in-time evolution equation, which is a familiar Bloch-Riedel equation for the density matrix modified by the counting field.[33] It reads:

$$\begin{aligned} \frac{\partial\hat{\rho}}{\partial t} &= -\frac{i}{\hbar}[\hat{H}_q,\hat{\rho}] + \frac{\chi^2}{2}S_{22}\hat{\rho} + \frac{ia_{12}\chi}{2}(\hat{\rho}\hat{\sigma}_z + \hat{\sigma}_z\hat{\rho}) \\ &\quad - \frac{S_{12}}{\hbar}\chi(\hat{\rho}\hat{\sigma}_z - \hat{\sigma}_z\hat{\rho}) - \frac{S_{11}}{\hbar^2}(\hat{\rho} - \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z). \end{aligned} \tag{5}$$

The locality in time is a relevant but strong assumption which in fact corresponds to a *classical* detector (indeed, the action (4) does not contain any $\hbar$.) This is why we do not have to worry about possible quantum uncertainties of the detector output that could complicate the interpretation of the statistics. [29] The scheme described can be easily extended to more qubits and/or detectors: One just adds extra (counting) fields for detectors and extra Pauli matrices for qubits. The case of interest for us is the simultaneous CWLM of three pseudo-spin projections. The coupling term becomes

$$H = \hat{\sigma}_1\hat{Q}_1 + \hat{\sigma}_2\hat{Q}_2 + \hat{\sigma}_3\hat{Q}_3 \,. \tag{6}$$

$\hat{Q}_k$ (k = 1, 2, 3) being the input fields of the three detectors . Three counting fields χ_k are coupled to the corresponding output variables V_k of the three detectors. We assume for simplicity that the detectors are independent and identical each described by the action (4). The corresponding Bloch-Reidel equation reads

$$\begin{aligned} \frac{\partial\hat{\rho}}{\partial t} &= -\frac{i}{\hbar}[\hat{H}_q,\hat{\rho}] + \frac{\chi^2}{2}S_{22}\hat{\rho} + \frac{ia_{12}}{2}\sum_{k=1}^{3}\chi_k[\hat{\sigma}_k,\hat{\rho}]_{+} \\ &\quad + \frac{S_{12}}{\hbar}\sum_{k=1}^{3}\chi_k[\hat{\sigma}_k,\hat{\rho}] - \frac{S_{11}}{\hbar^2}(3\hat{\rho} - \sum_{k=1}^{3}\hat{\sigma}_k\hat{\rho}\hat{\sigma}_k) \end{aligned} \tag{7}$$

where $\chi = \sqrt{\chi_1^2 + \chi_2^2 + \chi_3^2}$.

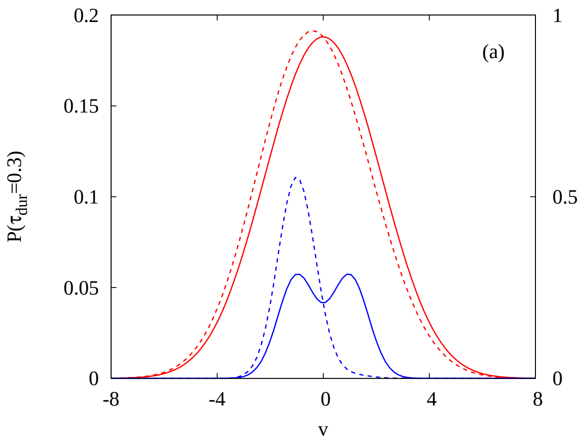

FIGURE 1. Quantum Non-demolition measurent: Two successive measurements. In each pair of the curves, the solid one gives the distribution of outcome of the first measurement while the dashed one gives the distribution for the second measurement *provided* the first measurement gave $v_1 = -1$. Lower (upper) pair of curves corresponds to long, $\tau_{1,2} = 2$ (short, $\tau_{1,2} = 0.3$) measurements. The long measurement is repetitive, the short one is not.

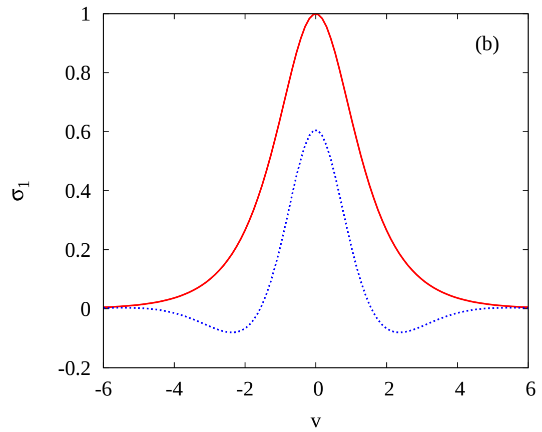

FIGURE 2. Quantum "undemolition" measurement: $\sigma_1(v,\tau)$ characterizes the dephasing of the superposition after a time $\tau(\tau = 1$ for plots) *provided* the detector outcome is v. A quantum-limited detector ($C = C_{12} = 0$, upper curve) allows for the quantum un-demolition measurement ($\sigma_1 = 1$) at $v = 0$. This does not work for a worse detector ($C = C_{12} = 1$, lower curve).

SINGLE VARIABLE

Let us first illustrate the method with one-detector QND measurement recently realized for superconducting qubits.[36] To satisfy non-demolishing condition [23], we should set $\hat{H}_q = \varepsilon\hat{\sigma}_3$. In this case, H_q is canceled by transformation to the rotating frame, $\rho(t) \to e^{i\hat{H}_q t/\hbar}\rho(t)e^{-i\hat{H}_q t/\hbar}$. Let us perform two measurements that immediately follow each other. During the first measurement, the detector output is collected in the time interval $(0,t_1)$ so the measurement outcome is $V_1 = \int_0^{t_1} dtV(t)/t_1$. Similarly, for the second measurement $V_2 = \int_{t_1}^{t_1+t_2} dtV(t)/t_2$. The statistics of the two outcomes is given by a piece-wise constant $\chi(t) = \chi_1(\chi_2)$ during the first(second) time interval and $\chi(t) = 0$ otherwise. We parameterize $\hat{\rho}$ as follows: $\hat{\rho}(t) = (1+\hat{\sigma}_3)\rho_+(t)/2 + (1-\hat{\sigma}_3)\rho_-(t)/2 + \hat{\sigma}_1\rho_1(t) + \hat{\sigma}_2\rho_2(t)$. Eq. (5) gives two decoupled pairs of equations for $\rho_\pm$ and $\rho_{1,2}$ respectively. Solving for $\rho_\pm(t)$ with initial conditions $\rho_\pm(0)$ (assuming $\rho_{1,2}(0) = 0$) and transforming the generating function gives a very simple probability distribution of two outcomes:

$$P(v_1,v_2) = \sum_\pm \frac{\sqrt{\tau_1\tau_2}}{2\pi}\rho_\pm(0)e^{-\frac{(v_1\mp1)^2\tau_1}{2}}e^{-\frac{(v_2\mp1)^2\tau_2}{2}}, \quad (8)$$

where we switched to the dimensionless times $\tau = t/\tau_m$, and outputs $v = V/a_{12}$. This result is in fact classical: It does not depend on the dephasing rate. Initially, the qubit comes either in the state $+$ or $-$ (with probabilities $\rho_\pm(0)$). The state persists during the measurements, the outcome of each measurement is distributed normally around ± 1 with the standard deviation $\sqrt{\tau_{1,2}}$. The repeatability of the measurements is illustrated in Fig. 1(a).

To illustrate the quantum aspect, let us set the initial wave function to a superposition: $\hat{\rho}(0) = \hat{\sigma}_1$, and evaluate the corresponding projection of the pseudo-spin *after* the measurement in time interval $(0,\tau)$ that gives the outcome v. In addition to equations for $\rho_\pm$, we have to solve two equations for $\rho_{1,2}$,

$$\frac{\partial\rho_{1,2}}{\partial t} = \mp\frac{2iS_{21}\chi}{\hbar}\rho_{2,1} - (\Gamma_d + \frac{\chi^2}{2}S_{22})\rho_{1,2}\,. \quad (9)$$

that do contain the dephasing rate. The quantity of interest is obtained by transforming the generating functions $\rho_1(\tau,\chi)$, $\mathrm{Tr}(\hat{\rho}(\tau,\chi))$ and reads (Fig. 2(b)):

$$\sigma_1(v,\tau) = \frac{\cos(C_{12}v\tau)}{\cosh(v\tau)}e^{-\frac{C}{2}\tau}, \quad (10)$$

where we introduce dimensionless constants $C \equiv 4(S_{11}S_{22} - S_{12}^2)/(\hbar a_{12})^2 - 1$ and $C_{12} = 2S_{12}/\hbar a_{12}$. Generally, $\sigma_1(v,\tau)$ quickly decays with increasing τ. This indicates the dephasing of the superposition by the measurement. Remarkably enough, for a quantum-limited detector ($C = C_{12} = 0$) and for a special value of the measurement outcome $v = 0$ the dephasing is absent and the wave function retains the initial value. This was called "quantum un-demolition measurement" in [10]. Physical meaning of this is that the phase shift between the states $\pm$, acquired from the detector, $2\int_0^\tau dtQ(t)/\hbar$, is zero at this (rather unprobable, [10]) value of the outcome. We stress that the strict correspondence between the phase shift and outcome does not hold for a general detector, so that $\sigma_1(v=0,\tau) = \exp(-C\tau/2)$ decreases with the time of measurement.

NON-COMMUTING VARIABLES

These simple examples prove the use of the statistical approach. Thus encouraged, we turn to the statistics of the CWLM of non-commuting variables. We assume that the Hamiltonian $\varepsilon\sigma_3$ is removed by transforming to the rotating frame. This presumes that the signal from $\sigma_{1,2}$ is collected at frequencies $\varepsilon/\hbar$ rather than at zero frequency as the signal from σ_3 is $V_{1,2} = \int_0^\tau dt(\cos(\varepsilon t/\hbar)V_{1,2}(t) \mp \sin(\varepsilon t/\hbar)V_{2,1}(t))/\tau$. Without the term H_q, the Eq. (7) is readily solved in proper basis in pseudo-spin space: One of the Pauli matrices is defined as $\hat{\sigma}_\chi = (\chi_1\hat{\sigma}_1 + \chi_2\hat{\sigma}_2 + \chi_3\hat{\sigma}_3)/\chi$, while two others are orthogonal to it.

We stress that the CWLM we are about to describe is hardly a measurement of the *initial* state of the qubit. In contrast to QND where the dephasing is limited to $1,2$ components, the detectors randomly rotate the pseudo-spin in all three directions. The quantum information about initial state is quickly lost at the time scale of $1/\Gamma_d$, that is, before a statistically reliable measurement result can be accumulated. To this end, the initial condition hardly matters and we choose the unpolarized density matrix $\hat{\rho}(0) = \frac{1}{2}\hat{1}$. Albeit we will see that this CWLM can be rather informative.

Let us first discuss the distribution of the detector outputs. In the limit of long time of measurement $\tau \gg 1$, the log of the generating function reads:

$$-\log Z = \tau\left(C_d - \sqrt{C_d^2 - \chi^2} + \frac{\chi^2}{2}\right). \qquad (11)$$

where χ_i has been made dimensionless $\chi_i S_{22}/a_{12} \to \chi_i$ as to give the cumulants of dimensionless outputs v_i. Here, $C_d \equiv \Gamma_d\tau_m = (C+1+C_{12}^2)/2 \geq 1/2$. The distribution is isotropic in three outputs depending on $v \equiv \sqrt{v_1^2+v_2^2+v_3^2}$ only and in this limit is obtained by the saddle-point method. The distribution is concentrated at zero and is essentially non-Gaussian at $v \simeq 1$. The presence of the qubit exponentially enhances probabilities of such outcomes.(Fig. 2(a)) As to low cumulants, the presence of the qubit enhances output noises of each detector by the factor $1+1/C_d$. Most importantly, it gives rise to non-zero fourth cumulant $\langle\langle v_i^2 v_j^2\rangle\rangle = -(1+2\delta_{ij})/(C_d\tau)^3$— correlation of noise *variations* in formally independent detectors.

Let us discuss the correlation of the detector outputs and the pseudo-spin *after* such measurement. We characterize this with a fidelity $f(v)$, inner product of the normalized vector of the outcomes and averaged pseudo-spin at *given* outcome v, $f = \sum_i \langle\sigma_i\rangle v_i/v$. The fidelity is 1 if the values of the outputs precisely give all three pseudo-spin components. From the saddle-point solution we obtain that f does not depend on τ in the limit $\tau \gg 1$, and at large values of the outcomes $v \gg 1$ reaches the ideal value $f \approx 1 - C_d/v$. This, quite unexpectedly, enables an efficient *quantum monitoring* of non-commuting variables. Roughly, one continuously measures the system and waits for sufficiently high values of the outputs. When this is achieved, the state of the qubit is known with any accuracy desired.

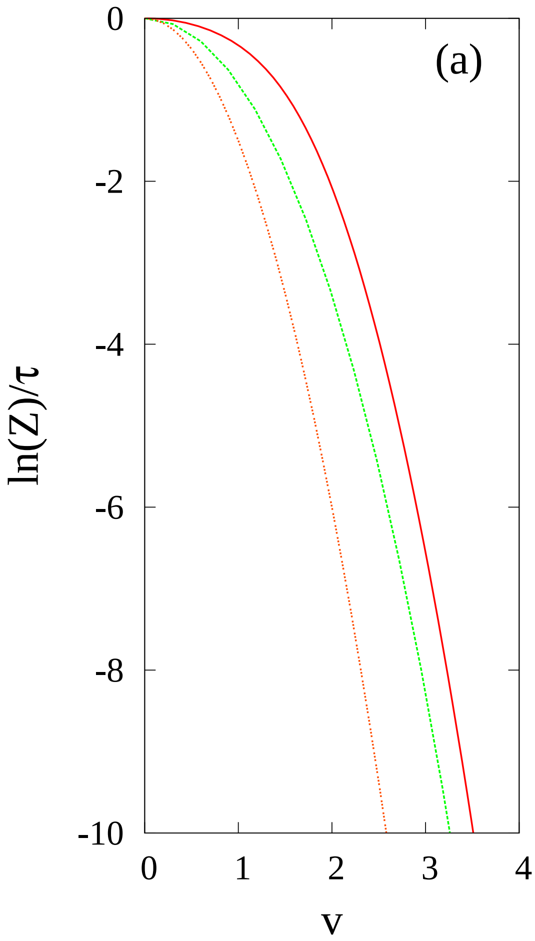

FIGURE 3. Statistics of CWLM of non-commuting variables. Logarithm of the outcome distribution is a function of $v \equiv \sqrt{v_1^2+v_2^2+v_3^2}$. The curves from the top to the bottom: quantum-limited detector ($C_d = 1/2$), worse detector ($C_d = 2$), detector not connected to the qubit. At big values of outputs $v \gg 1$, the probability of a big output is exponentially enhanced by the presence of the qubit.

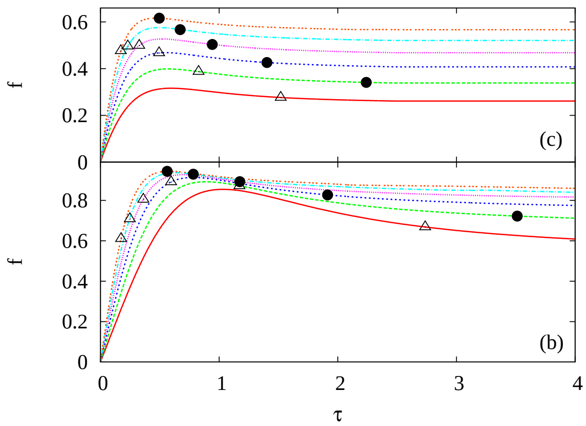

FIGURE 4. Fidelity of quantum monitoring f versus measurement time τ for v ranging from 4 to 1.5 (from upper to lower curves in each subfigure) Bullets (triangles) at each curve indicate the value of τ at which the probability to get the outcome $> v$ is 10% (50 %). (b) The quantum-limited detector. (c) A worse detector ($C_d = 2$).

The better the accuracy desired $a_{des} \equiv 1 - f \ll 1$, the bigger outputs are required, $v \gg C_d/a_{des}$, so the typical waiting time grows exponentially, $\log(t_w) \simeq a_{des}^{-2}$. To prove that the monitoring is practical, we have to show that a reasonably high fidelity can be achieved in a reasonably short time. We evaluate and plot (Fig. 2(b) and 2(c)) $f(v)$ of a single measurement of duration τ versus τ. We see from the plots that $f = 0.95$ is achieved for a quantum-limited detector at $v = 4$ and $\tau = 0.7$. At these parameters, 10% of the measurements are successful, i.e. give the output $v > 4$. We conclude that the 5% accuracy is typically achieved in time interval $\simeq 7\tau_m$.

To conclude, we present the statistical approach to the CWLM and illustrate the use of it. We propose a scheme for an efficient monitoring of non-commuting variables and prove its feasibility.

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