BFKL Pomeron and production amplitudes in $N = 4$ SUSY

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Abstract. Theoretical approaches to the problem of the high energy hadron-hadron scattering in the Regge kinematics are reviewed. It is shown, that the gluon in QCD is reggeized and the Pomeron is a two gluon composite state. Further, the equation for the multi-gluon composite states is integrable at $N_c \to \infty$. Due to the AdS/CFT correspondence in $N = 4$ SUSY the BFKL Pomeron is equivalent to the reggeized graviton. The important properties of the maximal transcendentality and integrability are realized in this model. Multi-gluon scattering amplitudes are investigated in the Regge limit. The BDS ansatz for them is not valid beyond one loop due to the presence of the Mandelstam cuts. The hamiltonian for the corresonding reggeon states coincides with the hamiltonian of an integrable open Heisenberg spin chain.

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REGGE APPROACH TO HIGH ENERGY INTERACTIONS

Hadron-hadron scattering in the Regge kinematics

$$s = (p_A + p_B)^2 = (2E)^2 >> \vec{q}^{\,2} = -(p_{A'} - p_A)^2 \sim m^2 \quad (1)$$

is usually described in terms of a t-channel exchange of the Reggeon

$$A_p(s,t) = \xi_p(t)\, g(t)\, s^{j_p(t)}\, g(t)\,,\; j_p(t) = j_0 + \alpha' t\,,\; \xi_p(t) = \frac{e^{-i\pi j_p(t)} + p}{\sin(\pi j_p)}\,, \quad (2)$$

where $j_p(t)$ is the Regge trajectory which is assumed to be linear, j_0 and α' are its itercept and slope, respectively. The signature factor ξ_p is a complex quantity depending on the Reggeon signature $p = \pm 1$. Around 40 years ago V.N. Gribov constructed the phenomenological field theory for all possible Reggeon interactions. A special Reggeon - Pomeron with the vacuum quantum numbers and positive signature was introduced to explain an approximately constant behavior of total cross-sections at high energies and a fullfillment of the Pomeranchuck theorem $\sigma_{h\bar{h}}/\sigma_{hh} \to 1$.

In the Born approximation of QCD the elastic amplitude for two colored particle scattering is factorized

$$M_{AB}^{A'B'}(s,t)|_{Born} = \Gamma^c_{A'A}\, \frac{2s}{t}\, \Gamma^c_{B'B}\,,\; \Gamma^c_{A'A} = g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\,, \quad (3)$$

where T^c are the generators of the color group $SU(N_c)$ in the corresponding representation and λ_r are helicities of the colliding and final state particles. In the leading logarithmic approximation (LLA) this amplitude has the Regge form [1]

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\, s^{\omega(t)},\; \alpha_s \ln s \sim 1\,, \quad (4)$$

where the gluon Regge trajectory with the use of the dimensional regularization can be written as follows

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{(2\pi)^{2-2\epsilon}}\, |q^2| \int \frac{\mu^{2\epsilon}\, d^{2-2\epsilon}k}{|k|^2|q-k|^2} \approx -a\left(\ln\frac{|q^2|}{\mu^2} - \frac{1}{\epsilon}\right),\; a = \frac{\alpha_s N_c}{2\pi}\left(4\pi\, e^{-\gamma}\right)^{\epsilon} \quad (5)$$

and γ is the Euler constant $\gamma = -\psi(1)$. This Regge trajectory was calculated also in two-loop approximation in QCD [2] and in supersymmetric gauge theories [3].

Further, the scattering amplitude in the multi-Regge kinematics for produced gluons with momenta k_r

$$s \gg s_1\,,\, s_2\,, \ldots,\, s_{n+1} \gg t_1\,,\, t_2\,, \ldots,\, t_{n+1}\,; \quad s_r = (k_{r-1} + k_r)^2\,,\; t_r = -|q_r|^2 \quad (6)$$

has the form [1]

$$M_{2\to 1+n} = 2s\, \Gamma^{c_1}_{A'A}\, \frac{s_1^{\omega(-|q_1|^2)}}{|q_1|^2}\, g T^{d_1}_{c_2 c_1} C(q_2,q_1) \times \frac{s_2^{\omega(-|q_2|^2)}}{|q_2|^2} \ldots C(q_n,q_{n-1})\, \frac{s_n^{\omega(-|q_n|^2)}}{|q_n|^2}\, \Gamma^{c_n}_{B'B}\,. \quad (7)$$

Here C are the Reggeon-Reggeon-gluon vertices

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*} \quad (8)$$

and we used the complex notations $k = k_x + ik_y$ for the transverse components of momenta.

BFKL EQUATION

Because the production amplitudes in QCD are factorized, one can write a Bethe-Salpeter-type equation for the total cross-section σ_t. It is governed by the Pomeron exchange. The Pomeron wave function satisfies the equation of Balitsky, Fadin, Kuraev and Lipatov (BFKL) [1]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\Delta = -\frac{\alpha_s N_c}{2\pi}\,E\,, \tag{9}$$

where $\sigma_t \sim s^{\Delta_{max}}$. The BFKL Hamiltonian in the coordinate representation ρ is

$$H_{12} = \ln|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\,, \tag{10}$$

where $\rho_{12} = \rho_1 - \rho_2$. It is invariant under the Möbius transformations [4, 5]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d} \tag{11}$$

and has the property of the holomorphic separability

$$H_{12} = h_{12} + h_{12}^*\,,\; h_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\ln(\rho_{12})\,p_1 + \frac{1}{p_2}\ln(\rho_{12})\,p_2 - 2\psi(1)\,. \tag{12}$$

Here we used the complex notations $\rho_r = x_r + iy_r$, $p_r = i\partial_r$ for two-dimensional transverse coordinates and their canonically conjugated momenta. For the principal series of unitary representations of the Möbius group the conformal weights are

$$m = \gamma + n/2\,,\; \widetilde{m} = \gamma - n/2\,,\; \gamma = 1/2 + i\nu\,, \tag{13}$$

where γ is the anomalous dimension of the twist-2 operators and $n = 0, \pm1, \pm2, ...$ is the conformal spin.

The Bartels-Kwiecinski-Praszalowicz (BKP) equation for colorless composite states of several reggeized gluons has the form [6]

$$E\,\Psi(\vec{\rho}_1,...\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...\vec{\rho}_n)\;,\; H = \sum_{k<l}\frac{\vec{T}_k\vec{T}_l}{-N_c}H_{kl}\,, \tag{14}$$

where H_{kl} is the BFKL hamiltonian. Apart from the Möbius invariance its wave function in the multi-color QCD ($N_c \to \infty$) has the property of the holomorphic factorization [7]

$$\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1,...,\rho_n)\,\Psi_s(\rho_1^*,...,\rho_n^*)\,, \tag{15}$$

where the sum is performed over a degenerate set of solutions for the corresponding holomorphic and anti-holomorphic equations. The BKP equation has also the duality symmetry [8]

$$p_k \to \rho_{k,k+1} \to p_{k+1} \tag{16}$$

and n integrals of motion q_r, q_r^* [9]. The corresponding hamiltonians h and h^* are local hamiltonians of the integrable Heisenberg spin model, in which spins are generators of the Möbiuos group [10]. We can introduce the transfer (T) and monodromy (t) matrices according to the definitions [9]

$$T(u) = Tr\,t(u) = \sum_{r=0}^{n} u^{n-r}\,q_r\,,\; t(u) = L_1(u)L_2(u)...L_n(u)\,, \tag{17}$$

$$L_k(u) = \begin{pmatrix} u+\rho_k p_k & p_k \\ -\rho_k^2 p_k & u-\rho_k p_k \end{pmatrix}, \quad t(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{18}$$

The matrix elements of $t(u)$ satisfy some bilinear commutation relations following from the Yang-Baxter equation [9]

$$t_{r_1'}^{s_1}(u)\,t_{r_2'}^{s_2}(v)\,\hat{l}_{r_1 r_2}^{r_1' r_2'}(v-u) = \hat{l}_{s_1' s_2'}^{s_1 s_2}(v-u)\,t_{r_2}^{s_2'}(v)\,t_{r_1}^{s_1'}(u)\,,\quad \hat{l}(u) = u\,\hat{1} + i\hat{P}\,, \tag{19}$$

where $\hat{l}(u)$ is the monodromy matrix for the usual Heisenberg spin model and $\hat{P}$ is the permutation operator. This equation can be solved with the use of the Bethe ansatz and the Baxter-Sklyanin approach [11, 12].

POMERON IN $N = 4$ SUSY

One can calculate the integral kernel for the BFKL equation also in two loops [13]. Its eigenvalue can be written as follows

$$\omega = 4\,\hat{a}\,\chi(n,\gamma) + 4\,\hat{a}^2\,\Delta(n,\gamma)\,,\; \hat{a} = g^2 N_c/(16\pi^2)\,, \tag{20}$$

where

$$\chi(n,\gamma) = 2\Psi(1) - \Psi(\gamma + |n|/2) - \Psi(1-\gamma+|n|/2)\,,\quad \Psi(x) = \Gamma'(x)/\Gamma(x)\,. \tag{21}$$

The one-loop correction $\Delta(n,\gamma)$ in QCD contains the non-analytic terms - the Kroniker symbols $\delta_{|n|,0}$ and $\delta_{|n|,2}$, but in $N=4$ SUSY they are cancelled and we obtain for $\Delta(n,\gamma)$ the following result [3, 14]

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,,$$
$$M = \gamma + \frac{|n|}{2}\,, \tag{22}$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\ \beta'(z)$$
$$= \frac{1}{4}\left[\Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right)\right]. \tag{23}$$

It is interesting, that all functions entering in these expressions have the property of the maximal transcendentality [14]. In particular, $\phi(M)$ can be written in the form

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big), \tag{24}$$
$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\ -\ \frac{\Psi(k+1)-\Psi(1)}{k+M}\right). \tag{25}$$

Here $\Psi(M)$ has the transcedentality equal to 1, its differentiation $\Psi^{(n)}$ increases it to $n+1$, the special number $\zeta(3)$ has the transcendality 3, the additional poles in the sum over k add the transcedentality of the function $\Phi(M)$ up to 3. The maximal transcendentality hypothesis is valid also for the anomalous dimensions of twist-2 -operators in $N=4$ SUSY [15, 16] contrary to the case of QCD [17].

The eigenvalue of the BFKL kernel in the diffusion approximation is written below [1]

$$j = 2 - \Delta - D\nu^2\,, \tag{26}$$

where ν is related to the anomalous dimension γ of the twist-2 operators as follows [13]

$$\gamma = 1 + \frac{j-2}{2} + i\nu\,. \tag{27}$$

The parameters Δ and D are functions of the coupling constant $\hat{a}$ and are known up to two loops. Higher order perturbative corrections can be obtained with the use of the effective action [18, 19]. For large coupling constants one can expect, that the leading Pomeron singularity in $N=4$ SUSY is moved to the point $j=2$ and asymptotically the Pomeron coincides with the graviton Regge pole. This assumption is related to the AdS/CFT correspondence, formulated in the framework of the Maldacena hypothesis claiming, that $N=4$ SUSY is equivalent to the superstring model living on the 10-dimensional anti-de-Sitter space [20, 21, 22]. Therefore it is natural to impose on the BFKL equation in the diffusion approximation the physical condition, that for the conserved energy-momentum tensor $\theta_{\mu\nu}(x)$ having $j=2$ the anomalous dimension γ is zero. As a result, we obtain, that the parameters Δ and D coincide [16] and

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right). \tag{28}$$

Using the dictionary developed in the framework of the AdS/CFT correspondence [21], one can rewrite the eigenvalue relation for the BFKL kernel in the form of the graviton Regge trajectory [16]

$$j = 2 + \frac{\alpha'}{2}t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\Delta\,. \tag{29}$$

On the other hand, Gubser, Klebanov and Polyakov predicted the following asymptotics of the anomalous dimension at large $\hat{a}$ and j [23]

$$\gamma_{|\hat{a},j\to\infty} = -\sqrt{2\pi j}\,\hat{a}^{1/4}\,. \tag{30}$$

As a result, one can obtain the explicit expression for the Pomeron intercept at large coupling constants [16, 24]

$$j = 2 - \Delta\,,\ \Delta = \frac{1}{2\pi}\,\hat{a}^{-1/2}\,. \tag{31}$$

In Ref. [25] it was argued, that for $N=4$ SUSY the evolution equations for anomalous dimensions of quasi-partonic operators are integrable in LLA. Later such integrability was generalized to other operators [26] and to higher loops [27]. Using additionally the maximal transcendentality hypothesis the integral equation for the so-called casp anomalous dimension was constructed in all orders of perturbation theory [28, 29]. Further, the anomalous dimension of twist-2 operators in four loops was calculated [30], but due to the absence of the so-called wrapping contributions in the asymptotic Bethe anzatz the obtained results do not agree with the BFKL predictions [3, 14].

BERN-DIXON-SMIRNOV SCATTERING AMPLITUDES IN $N=4$ SUSY

To calculate higher order corrections to the BFKL equation in QCD and supersymmetric models one should know production amplitudes in higher orders of perturbation theory. Several years ago Bern, Dixon and Smirnov suggested a simple anzatz for the multi-gluon scattering amplitude with the maximal helicity violation in the planar limit $\alpha N_c \sim 1$ for the $N=4$ super-symmetric gauge theory [31]. It turns out, that this amplitude is proportional to its Born expression. The proportionality coefficient M_n for n external particles is a function of relativistic invariants and can be expressed at $\varepsilon = (4-D)/2 \to 0$ in terms of an infraredly divergent factor and an expression depending on three functions $\gamma(a), \beta(a)$ and $\delta(a)$, which are known up to a rather large order of perturbation theory. In particular, $\gamma(a)$ is the so-called cusp anomalous dimension which was calculated in all orders [28, 29]

In Ref. [32] the BDS anzatz was investigated in the Regge kinematics (see also Ref. [33]). In particular, the elastic amplitude has the Regge asymptotics

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)\,(1+\mathscr{O}(\varepsilon)), \quad (32)$$

where μ^2 is the renormalization point. The quantity

$$\omega(t) = -\frac{\gamma(a)}{4}\ln\frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'}\left(\frac{\gamma(a')}{4\varepsilon} + \beta(a')\right)$$
$$= \left(-\ln\frac{-t}{\mu^2} + \frac{1}{\varepsilon}\right)a + \left[\zeta_2\left(\ln\frac{-t}{\mu^2} - \frac{1}{2\varepsilon}\right) - \frac{\zeta_3}{2}\right]a^2 + \ldots \quad (33)$$

is the all-order gluon Regge trajectory obtained from the BDS formula [32] and

$$\ln\Gamma(t) = \ln\frac{-t}{\mu^2}\int_0^a \frac{da'}{a'}\left(\frac{\gamma(a')}{8\varepsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma(a)}{2}\zeta_2 - \int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma(a')}{4\varepsilon^2} + \frac{\beta(a')}{\varepsilon} + \delta(a')\right), \quad (34)$$

is the vertex for the Reggeized gluon coupling to the external particles. Note that the perturbative expansion for $\omega(t)$ is in an agreement with its direct calculations performed initially in the $\overline{MS}$-scheme [3].

One can verify that in all physical regions the BDS amplitude for one gluon production in the multi-Regge kinematics can be obtained with the use of an analytic continuation from the expression [32]

$$\frac{M_{2\to 3}}{\Gamma(t_1)\Gamma(t_2)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)}\left(\frac{-s\kappa}{\mu^4}\right)^{\omega(t_2)} c_1 + \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)-\omega(t_1)}\left(\frac{-s\kappa}{\mu^4}\right)^{\omega(t_1)} c_2, \quad (35)$$

where $\kappa = s_1 s_2/s = |k_\perp|^2$ and the coefficients c_i are real

$$c_1(\kappa) = |\Gamma(t_2,t_1,\ln -\kappa)|\,\frac{\sin\pi(\omega(t_1)-\phi_\Gamma)}{\sin\pi(\omega(t_1)-\omega(t_2))} \quad (36)$$

$$c_2(\kappa) = |\Gamma(t_2,t_1,\ln -\kappa)|\,\frac{\sin\pi(\omega(t_2)-\phi_\Gamma)}{\sin\pi(\omega(t_2)-\omega(t_1))} \quad (37)$$

Here ϕ_Γ is the phase of the Reggeon-Reggeon-gluon vertex Γ, *i.e.*

$$\Gamma(t_2,t_1,\ln\kappa - i\pi) = |\Gamma(t_2,t_1,\ln -\kappa)|\,e^{i\pi\phi_\Gamma}, \quad (38)$$

$$\ln\Gamma(t_2,t_1,\ln -\kappa) = -\frac{\gamma(a)}{16}\ln^2\frac{-\kappa}{\mu^2} - \frac{1}{2}\int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma(a')}{4\varepsilon^2} + \frac{\beta(a')}{\varepsilon} + \delta(a')\right) - \frac{\gamma(a)}{16}\ln^2\frac{-t_1}{-t_2} - \frac{\gamma(a)}{16}\zeta_2 - \frac{1}{2}\left(\omega(t_1)+\omega(t_2) - \int_0^a \frac{da'}{a'}\left(\frac{\gamma(a')}{4\varepsilon} + \beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2}. \quad (39)$$

In a similar way two gluon production amplitude in the multi-Regge kinematics almost in all physical regions can be obtained by an analytic continuation from a dispersion-like representation containing 5 contributions. However, in the physical kinematical region, where $s, s_2 > 0$ but $s_1, s_3 < 0$ the Regge factorization for the BDS amplitude is broken [32]

$$\frac{M_{2\to 4}}{\Gamma(t_1)\Gamma(t_3)} = C\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1,\ln -\kappa_{12}) \times \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2,\ln -\kappa_{23})\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}, \quad (40)$$

where the coefficient C is given below

$$C = \exp\left[\frac{\gamma_K(a)}{4}\,i\pi\left(\ln\frac{\vec{q}_1^{\,2}\vec{q}_3^{\,2}}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\varepsilon}\right)\right]. \quad (41)$$

Similarly for the BDS amplitude describing the transition $3\to 3$ in the physical region, where $s, s_2 = t_2' > 0$ but $s_1, s_3 < 0$ we obtain the result

$$\frac{M_{3\to 3}}{\Gamma(t_1)\Gamma(t_3)} = C'\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1,\ln -\kappa_{12}) \times \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_2,t_1,-\ln\kappa_{23})\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}, \quad (42)$$

where the phase factor C' is

$$C' = \exp\left[\frac{\gamma_K(a)}{4}(-i\pi)\ln\frac{(\vec{q}_1-\vec{q}_2)^2(\vec{q}_2-\vec{q}_3)^2}{(\vec{q}_1+\vec{q}_3-\vec{q}_2)^2\vec{q}_2^{\,2}}\right], \quad (43)$$

which also contradicts the Regge factorization. The reason for these drawbacks is that just in these kinematical regions the amplitudes $A_{2\to 4}$ and $A_{3\to 3}$ should contain the Mandelstam cuts in the j-pane of the t_2-channel [32]. Therefore the BDS amplitudes for these processes are not correct beyond 1 loop.

MANDELSTAM CUTS IN THE ADJOINT REPRESENTATION AT LLA

The Mandelstam cuts in the elastic amplitude appear only in the non-planar diagrams because the integrals over the Sudakov variables $\alpha = 2kP_A/s$ and $\beta = 2kp_B$ for the reggeon momenta k and $q-k$ should have the singularities above and below the corresponding integration contours. For the case of planar diagrams this Mandelstam condition is fulfilled only for inelastic amplitudes starting from six external particles in the kinematical region where $s, s_2 > 0$ and $s_1, s_3 < 0$. Two reggeons in the t_2-channel with an adjoint representation of the gauge group $SU(N_c)$ can also scatter each from another. The corresponding contribution to the imaginary part in the s_2-channel for the amplitude $A_{2\to4}$ can be written as follows [32]

$$\frac{1}{\pi}\Im_{s_2} M_{2\to4} = s_2^{\omega(t_2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \times\frac{d\omega}{2\pi i}\left(\frac{s_2}{\mu^2}\right)^{\omega} \widetilde{f}_2(\omega) \qquad (44)$$

The reduced partial wave $\widetilde{f}_2(\omega)$ is given by

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\varepsilon\, |q_2|^2 \int d^{2-2\varepsilon}k\, d^{2-2\varepsilon}k' \times\Phi_1(k,q_2,q_1)\, G_\omega(k,k',q_2)\, \Phi_3(k',q_2,q_3) \qquad (45)$$

where $\Phi_{1,3}$ are the impact factors

$$\Phi_1(k,q_2,q_1) = \frac{k_1^*(q_2-k)^*}{q_2^*(k+k_1)^*}\,, \qquad \Phi_3(k',q_2,q_3) = \frac{k_2(k'-q_2)}{q_2(k'-k_2)}\,. \qquad (46)$$

The Green's function $G_\omega(k,k',q_2)$ satisfies the BFKL-type equation

$$\omega G_\omega^{(8_A)}(k,k',q_2) = \frac{(2\pi)^3\delta^{(2)}(k-k')}{|k|^2|k+q_2|^2} + \frac{1}{|k|^2|k+q_2|^2}\left(K^{(8_A)}\otimes G_\omega^{(8_A)}\right)(k,k',q_2)\,, \qquad (47)$$

where

$$K^{(8_A)}(k,k';q_2) = \delta^{(2)}(k-k')\left(\omega(-|k|^2)+\omega(-|q_2-k|^2)-2\omega(-|q|^2)\right) + \frac{a}{2}\frac{k^*(q_2-k)k'(q_2-k')^*+c.c.}{|k-k|^2}\,. \qquad (48)$$

The infrared divergencies are extracted from $M_{2\to4}$ in the form of the Regge factor $s_2^{\omega(t_2)}$ and coincide with those of the BDS amplitude, as it should be. The partial wave $\widetilde{f}_2(\omega)$ contains the infrared divergency only in one loop

$$\hat{\alpha}_\varepsilon\, |q_2|^2 \int d^{2-2\varepsilon}k\, \frac{k^* q_1^*}{q_2^*(k+k_1)^*}\; \frac{1}{|k|^2|q_2-k|^2}\; \frac{kq_3}{q_2(k-k_2)} = \frac{a}{2}\left(\ln\frac{|q_1|^2|q_3|^2}{|k_1+k_2|^2\mu^2} - \frac{1}{\varepsilon}\right), \qquad (49)$$

which is also compatible with the BDS result. But in upper loops the iteration of the above equation leads to terms which are absent in the BDS amplitude. For example, in two loops we obtain for the imaginary part of $A_{2\to4}$ in the s_2-channel the more complicated expression [34]

$$A_{s_2} = \frac{a^2}{2}\ln s_2 \ln\frac{|q_1-q_3|^2|q_2|^2}{|q_1|^2|k_2|^2}\ln\frac{|q_1-q_3|^2|q_2|^2}{|q_3|^2|k_1|^2}\,. \qquad (50)$$

It is symmetric with respect to the simultaneous transmutation of momenta

$$k_1 \leftrightarrow k_2\,,\; q_1 \leftrightarrow -q_3\,. \qquad (51)$$

The same expression is valid also for the imaginary part in the s-channel.

In a similar way we can calculate the s-channel imaginary part of the amplitude for the transition $3 \to 3$

$$A_s^{3\to3} = \frac{a^2}{2}\ln t_2' \ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|k_1|^2|k_2|^2}\ln\frac{|q_2-q_1-q_3|^2|q_2|^2}{|q_3|^2|q_1|^2}\,. \qquad (52)$$

Moreover, the BFKL equation for the state with adjoint quantum numbers can be solved exactly and we obtain for the imaginary part in s_2-channel [34]

$$\Im M_{2\to4} \sim s_2^{\omega(t_2)} \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}} \times\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}\exp\left(\omega(\nu,n)\ln s_2\right)\,, \qquad (53)$$

where the eigenvalue of the reduced BFKL kernel for the adjoint representation is

$$\omega(\nu,n) = -a\left(\psi(i\nu+\frac{|n|}{2})+\psi(-i\nu+\frac{|n|}{2})-2\psi(1)\right). \qquad (54)$$

It turns out, that the leading singularity of the t_2-partial wave corresponds to $n=1$ and is situated at

$$j-1 = \omega(t_2) + a(4\ln 2 - 2).$$

MULTI-REGGEON MANDELSTAM CUTS

Let us consider now the Mandelstam cuts constructed from several reggeons [35]. The non-vanishing contribution from the exchange of $n+1$ reggeons appears in the planar diagrams only if the number of external lines is $r \geq 2n+4$. In the case of production of $2n$ gluons with the same helicity the amplitude in $N=4$ SUSY is proportional to the Born expression. In the leading logarithmic approximation for the $n+1$-reggeon contribution to the s_{n+1}-channel the proportionality factor has the form [35]

$$f_{LLA}^{2\to 2+2n} = \left(i\,\frac{g^2 N_c}{4\pi}\right)^n Q^* \widetilde{Q}$$
$$\int \prod_{l=1}^{n} \frac{\mu^{2\varepsilon} d^{2-2\varepsilon} p_l}{(2\pi)^{1-2\varepsilon}} \frac{\mu^{2\varepsilon} d^{2-2\varepsilon} p'_l}{(2\pi)^{1-2\varepsilon}}$$
$$\times \prod_{l=1}^{n} \frac{k_l^* k_{2r-l}}{|p_l|^2} \frac{G(p,p';s_{n+1})}{|p_{n+1}|^2} \Phi_1 \Phi_2 , \qquad (55)$$

where Q and $\widetilde{Q}$ are momentum transfers between momenta $p_A, p_{A'}$ and $p_B, p_{B'}$. The impact factors are

$$\Phi_1(\vec{p}_1, ..., \vec{p}_{n+1}) =$$

$$\prod_{l=1}^{n} \frac{p_{l+1}^*}{(Q^* - \sum_{s=1}^{l} p_s^* - \sum_{s=1}^{l-1} k_s^*)(Q^* - \sum_{t=1}^{l+1} p_t^* - \sum_{t=1}^{l-1} k_t^*)},$$

$$\Phi_2(\vec{p}'_1, ..., \vec{p}'_{n+1}) =$$

$$\prod_{l=1}^{n} \frac{p'_{l+1}}{(\widetilde{Q} + \sum_{s=1}^{l} p'_s - \sum_{s=1}^{l-1} k_{2n-s+1})(\widetilde{Q} + \sum_{t=1}^{l+1} p'_t - \sum_{t=1}^{l-1} k_{2n-t+1})}.$$

The multi-reggeon Green function satisfies the equation [35]

$$\frac{\partial}{\partial \ln s_{n+1}} G(\vec{p}, \vec{p}'; s_{n+1})$$
$$= K\, G(\vec{p}, \vec{p}'; s_{n+1}) ,$$
$$G(\vec{p}, \vec{p}'; 0) = \prod_{l=1}^{n} \frac{(2\pi)^{1-2\varepsilon}}{\mu^{2\varepsilon}} \delta^{2-2\varepsilon}(p_l - p'_l) . \qquad (56)$$

Here the kernel K in LLA can be expressed in terms of the infraredly stable Hamiltonian H

$$K = \omega(t) - \frac{g^2 N_c}{16\pi^2} H , \; \omega(t) = a\left(\frac{1}{\varepsilon} - \ln\frac{-t}{\mu_2}\right) , \; t = -|q|^2 , \qquad (57)$$

$$H = \ln \frac{|p_1|^2 |p_{n+1}|^2}{|q|^4} + \sum_{l=1}^{n} H_{l,l+1} , \qquad (58)$$

where the pair Hamiltonians are

$$H_{l,l+1} = \ln |p_l|^2 + \ln |p_{l+1}|^2$$
$$+ p_l\, p_{l+1}^* \ln |\rho_{l,l+1}|^2 \frac{1}{p_l\, p_{l+1}^*}$$
$$+ p_l^*\, p_{l+1} \ln |\rho_{l,l+1}|^2 \frac{1}{p_l^*\, p_{l+1}} . \qquad (59)$$

INTEGRABLE OPEN HEISENBERG SPIN CHAIN

The Hamiltonian for the $n+1$-gluon composite state in the ajoint representation has the property of the holomorphic separability [35]

$$H = h + h^* , \; h = \ln \frac{p_1\, p_{n+1}}{q^2} + \sum_{l=1}^{n} h_{l,l+1} , \qquad (60)$$

where

$$h_{l,l+1} = \ln p_l + \ln p_{l+1} + p_l \ln \rho_{l,l+1} \frac{1}{p_l} + p_{l+1} \ln \rho_{l,l+1} \frac{1}{p_{l+1}} . \qquad (61)$$

Using the duality transformations (cf. [8])

$$p_1 = z_{0,1} , \; p_r = z_{r-1,r} , \; q = z_{0,n} , \; \rho_{r,r+1} = i \frac{\partial}{\partial z_r} = i\partial_r , \qquad (62)$$

the holomorphic hamiltonian can be rewritten in the form invariant under the Möbius transformations

$$z_k \to \frac{a z_k + b}{c z_k + d} . \qquad (63)$$

Therefore we can put

$$z_0 = 0 , \; z_n = \infty , \qquad (64)$$

Further, by regrouping the terms one can present it in another form [35]

$$\ln(z_1^2 \partial_1) + \ln(\partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h'_{r,r+1} , \qquad (65)$$

where

$$h'_{r,r+1} = \ln(z_{r,r+1}^2 \partial_r) + \ln(z_{r,r+1}^2 \partial_{r+1}) - 2 \ln z_{r,r+1} + 2\gamma$$
$$= \ln(\partial_r) + \ln(\partial_{r+1}) + \frac{1}{\partial_r} \ln z_{r,r+1}\, \partial_r + \frac{1}{\partial_{r+1}} \ln z_{r,r+1}\, \partial_{r+1} + 2\gamma . \qquad (66)$$

The pair hamiltonian $h'_{r,r+1}$ coincides in fact with the expression (12) in the coordinate representation acting on the wave function with non-amputated propagators.

The remarkable property of h is its commutativity with the matrix element $D(u)$ of the monodromy matrix (18) introduced above for the description of integrability of the BKP equations in the multi-color QCD [35]

$$[D(u), h] = 0 . \qquad (67)$$

Therefore if we write $D(u)$ as a polynomial in u

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k , \qquad (68)$$

then the differential operators

$$q'_k = -\sum_{0<r_1<r_2<\ldots<r_k<n} z_{r_1} \prod_{s=1}^{k-1} z_{r_s,r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t} \quad (69)$$

are independent integrals of motion with the properties

$$[q'_k, h] = [q'_k, q'_t] = 0\,. \quad (70)$$

It turns out, that h coincides with the local hamiltonian of the open integrable Heisenberg model in which spins are generators of the Möbius group.

To solve this model one can use the algebraic Bethe anzatz. In this case it is convenient to go to the transposed space, where there exists the pseudo-vacuum state Ψ_0

$$\Psi_0 = \prod_{r=1}^{n-1} z_r^{-2}\,, \quad (71)$$

satisfying the equation

$$C^t(u)\Psi_0 = 0\,. \quad (72)$$

Here $C^t(u)$ is the transposed matrix element $C(t)$ of the monodromy matrix (18). The eigenvalues of the hamiltonian and the integral of motion $D(u)$ are constructed by applying the product of its matrix elements $B^t(u)$ to the pseudovacuum state

$$\Psi_k = \prod_{r=1}^{k} B^t(u_r)\,\Psi_0\,. \quad (73)$$

For such eigenfunctions the spectral parameters u_r should obey the Bethe equations. Instead one can introduce the Baxter function which is the generating function of the Bethe roots

$$Q(u) = \prod_{k=1}^{\infty} (u - u_k)\,. \quad (74)$$

Generally the number of the roots u_k is infinite. The Baxter function satisfies the Baxter equation which is reduced to the simple recurrent relation for our open spin chain

$$\Lambda(u)\,Q(u) = (u+i)^{n-1}\,Q(u+i)\,, \quad (75)$$

where $\Lambda(u)$ is an eigenvalue of the integral of motion $D(u)$ and can be written in terms of its roots

$$D(u)\,\Psi_{a_1,a_2,\ldots,a_{n-1}} = \Lambda(u)\,\Psi_{a_1,a_2,\ldots,a_{n-1}}\,,\ \Lambda(u) = \prod_{r=1}^{n-1}(u - ia_r)\,. \quad (76)$$

As a result, the solution of the Baxter equation can be found in the form [35]

$$Q(u) = \prod_{r=1}^{n-1} \frac{\Gamma(-iu - a_r)}{\Gamma(-iu+1)} \quad (77)$$

up to a possible factor being a periodic function of $-iu$.

The Regge trajectory of the composite state of $n-1$ gluons has the additivity property

$$\omega_n(t) = \omega(t) - \frac{a}{2}E\,,\ E = \sum_{r=1}^{n-1} \varepsilon(a_r) + \sum_{r=1}^{n-1} \varepsilon(\widetilde{a}_r)\,, \quad (78)$$

where

$$\varepsilon(a) = \psi(a) + \psi(1-a) - 2\psi(1)\,,\ a_r = i\nu_r + \frac{n_r}{2}\,. \quad (79)$$

THREE GLUON COMPOSITE STATE

The wave function of the three gluon composite state in the adjoint representation can be constructed as a bilinear combination of eigenfunctions of the integrals of motion $D(u)$ and $D^*(u)$ having the property of single-valuedness in the coordinate space [35]

$$\Psi \sim z_2^{a_1+a_2}\,(z_2^*)^{\widetilde{a}_1+\widetilde{a}_2}$$
$$\int \frac{d^2y}{|y|^2}\, y^{-a_2}(y^*)^{-\widetilde{a}_2} \left(\frac{y-1}{y-x}\right)^{a_1} \left(\frac{y^*-1}{y^*-x^*}\right)^{\widetilde{a}_1},$$
$$x = \frac{z_2}{z_1}.$$

One can perform its Fourie transformation to the momentum space

$$\Psi^t(\vec{p}_1,\vec{p}_2) = (p_1+p_2)^{-a_1-a_2}(p_1^*+p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2}\,\phi(\vec{y})\,,\ y = \frac{p_2}{p_1}\,,$$

where

$$\phi(\vec{y}) = \int d^2t \left(\frac{1}{t\,y}+1\right)^{a_1} \left(\frac{1}{t^*y^*}+1\right)^{\widetilde{a}_1} (1-t)^{a_2-1}\,(1-t^*)^{\widetilde{a}_2-1}\,.$$

This function can be presented in terms of its Mellin transformation

$$\Psi^t(\vec{p}_1,\vec{p}_2) = (p_1+p_2)^{-a_1-a_2}(p_1^*+p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2}$$
$$\int d^2u\,\phi(u,\widetilde{u}) \left(\frac{p_1}{p_2}\right)^{-iu} \left(\frac{p_1^*}{p_2^*}\right)^{-i\widetilde{u}},$$

where

$$-iu = i\nu_u + \frac{N_u}{2}\,,\ -i\widetilde{u} = i\nu_u - \frac{N_u}{2}\,,\ \int d^2u \equiv \int_{-\infty}^{\infty} d\nu_u \sum_{N_u=-\infty}^{\infty}\,.$$

and

$$\phi(u,\widetilde{u}) = \frac{\pi^2\Gamma(1+\widetilde{a}_1)\Gamma(a_2)}{\Gamma(-a_1)\Gamma(1-\widetilde{a}_2)}\,\frac{\Gamma(iu)\Gamma(1+i\widetilde{u})}{\Gamma(-iu)\Gamma(1-i\widetilde{u})}$$
$$\frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma(1+i\widetilde{u}+\widetilde{a}_1)\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}.$$

Really the last form of Ψ^t corresponds to the Baxter-Sklyanin representation [11], because the function ϕ is a product of the pseudovacuum state and the Baxter function [35]

$$\phi(u,\widetilde{u}) = u\widetilde{u}\,Q(u,\widetilde{u})\,,$$

where

$$Q(u,\widetilde{u}) \sim \frac{\Gamma(iu)\Gamma(i\widetilde{u})}{\Gamma(1-iu)\Gamma(1-i\widetilde{u})} \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma(1+i\widetilde{u}+\widetilde{a}_1)\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}.$$

DISCUSSION OF OBTAINED RESULTS

It was demonstated, that Pomeron in QCD is a composite state of reggeized gluons. The BFKL dynamics is integrable in LLA. In the next-to-leading approximation in $N = 4$ SUSY the equation for the Pomeron wave function has remarkable properties including the analyticity in the conformal spin n and the maximal transcendentality. In this model the BFKL Pomeron coincides with the reggeized graviton. The BDS ansatz for scattering amplitudes in $N = 4$ SUSY does not agree with the BFKL approach in the multi-Regge kinematics. The reason for this drawback is the absence of the Mandelstam cuts. The BFKL-like equation for the composite state of two reggeized gluons with adjoint quantum numbers is explicitely solved. It is shown, that the equation for the composite state of an arbitrary number of reggeized gluons in the adjoint representation is equivalent to the Schrödinger equation for an integrable open Heisenberg spin chain. The wave function for three gluon composite state is constructed in the Baxter-Sklyanin representation.

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