Bethe Ansatz and supersymmetric vacua

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Abstract. Supersymmetric vacua of two dimensional $\mathcal{N}=4$ gauge theories with matter, softly broken by the twisted masses down to $\mathcal{N}=2$, are shown to be in one-to-one correspondence with the eigenstates of integrable spin chain Hamiltonians. Examples include: the Heisenberg $SU(2)$ XXX spin chain which is mapped to the two dimensional $U(N)$ theory with fundamental hypermultiplets, the XXZ spin chain which is mapped to the analogous three dimensional super-Yang-Mills theory compactified on a circle, the XYZ spin chain and eight-vertex model which are related to the four dimensional theory compactified on $\mathbf{T}^2$. A consequence of our correspondence is the isomorphism of the quantum cohomology ring of various quiver varieties, such as cotangent bundles to (partial) flag varieties and the ring of quantum integrals of motion of various spin chains. The correspondence extends to any spin group, representations, boundary conditions, and inhomogeneity, it includes Sinh-Gordon and non-linear Schrödinger models as well as the dynamical spin chains like Hubbard model. Compactifications of four dimensional $\mathcal{N}=2$ theories on a two-sphere lead to the instanton-corrected Bethe equations.

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GAUGE THEORIES AND INTEGRABLE SYSTEMS

The dynamics of gauge theory is a subject of long history and the ever growing importance.

In the last fifteen years or so it has become clear that the gauge theory dynamics in the vacuum sector is related to that of quantum many-body systems. A classic example is the equivalence of the pure Yang-Mills theory with gauge group $U(N)$ in two dimensons to the system of N free non-relativistic fermions on a circle. The same theory embeds as a supersymmetric vacuum sector of a (deformation of) $\mathcal{N}=2$ super-Yang-Mills theory in two dimensions.

A bit less trivial example found in [1] is that the vacuum sector of a certain supersymmetric two dimensional $U(N)$ gauge theory with massive adjoint matter is described by the solutions of Bethe ansatz equations for the quantum Nonlinear Schrödinger equation (NLS) in the N-particle sector. The model of [1] describes the $U(1)$-equivariant intersection theory on the moduli space of solutions to Hitchin's equations [2], just as the pure Yang-Mills theory describes the intersection theory on moduli space of flat connections on a two dimensional Riemann surface. This subject was revived in [3, 4] by showing that the natural interpretation of the results of [1] is in terms of the equivalence of the vacua of the $U(N)$ Yang-Mills-Higgs theory in a sense of [3] and the energy eigenstates of the N-particle Yang system, i.e. a system of N non-relativistic particles on a circle with delta-function interaction. Furthermore [3, 4] suggested that such a correspondence should be a general property of a larger class of supersymmetric gauge theories in various space-time dimensions.

Prior to [1] a different connection to spin systems with long-range interaction appeared in two dimensional pure Yang-Mills theory with massive matter [5, 6, 7, 8]. Three dimensional lift of latter gauge theory describes relativistic interacting particles [9], while four dimensional theories lead to elliptic generalizations [10].

In this paper we formulate precisely the correspondence between the two dimensional $\mathcal{N}=2$ supersymmetric gauge theories and quantum integrable systems in a very general setup. The $\mathcal{N}=2$ supersymmetric theories have rich algebraic structure surviving quantum corrections [11]. In particular, there is a distinguished class of operators $(\mathcal{O}_A)$, which commute with some of the nilpotent supercharges $\mathcal{Q}$ of the supersymmetry algebra. They have no singularities in their operator product expansion and, when considered up to the $\mathcal{Q}$-commutators, form a (super)commutative ring, called the chiral ring [11, 12]. The supersymmetric vacua of the theory form a representation of that ring. The space of supersymmetric vacua is thus naturally identified with the space of states of a quantum integrable system, whose Hamiltonians are the generators of the chiral ring. The duality states that the spectrum of the quantum Hamiltonians coincides with the spectrum of the chiral ring. The nontrivial result of this paper is that arguably all quantum integrable lattice models from the integrable systems textbooks correspond in this fashion to the $\mathcal{N}=2$ supersymmetric *gauge* theories, essentially also from the (different) textbooks. More precisely, the gauge theories which correspond to the integrable spin chains and their limits (the non-linear Schrödinger equation and other systems encountered in [1, 3, 4] being particular large spin limits thereof) are the softly broken $\mathcal{N}=4$ theories. It is quite important that we are dealing here with the gauge theories, rather then the general $(2,2)$ models, since it is in the gauge theory context that the equations describing the supersymmetric vacua can be identified with Bethe equations of the integrable world.

At this point we should clarify a possible confusion about the rôle of integrable systems in the description of the dynamics of supersymmetric gauge theories.

It is known that the low energy dynamics of the four dimensional $\mathcal{N}=2$ supersymmetric gauge theories is governed by the classical algebraic integrable systems [13]. Moreover, the natural gauge theories lead to integrable systems of Hitchin type, which are equivalent to many-body systems [14] and conjecturally to spin chains [15, 16, 17, 18].

We emphasize, however, that the correspondence between the gauge theories and integrable models we discuss in the present paper and in [1, 3, 4, 19, 20] is of a different nature. The low energy effective theory in four dimensions is described by the classical algebraic integrable systems of type [13], while the vacuum states we discuss presently are mapped to the quantum eigenstates of a different, quantum integrable system[1].

The gauge theories we study in two dimensions, as well as their string theory realizations, have a natural lift to three and four dimensions, while keeping the same number of supersymmetries, modulo certain anomalies. Indeed, the $\mathcal{N}=2$ super-Yang-Mills theory in two dimensions is a dimensional reduction of the $\mathcal{N}=1$ four dimensional Yang-Mills theory (this fact is useful in the superspace formulation of the theory [23]). Instead of the dimensional reduction one can take the compactification on a two dimensional torus $\mathbf{T}^2$. That way the theory will look macroscopically two dimensional, but its effective low energy dynamics is different due to the contribution of the Kaluza-Klein modes (the early examples of these corrections in the analogous compactifications from five to four dimensions can be found in [24]). This is seen, for example, in the geometry of the (classical) moduli space of vacua, which is compact for the theory obtained by compactification from four to two dimensions (it is isomorphic to the moduli space Bun_G of holomorphic $G_{\mathbf{C}}$-bundles on elliptic curve), and is noncompact in the dimensionally reduced theory. Quantum mechanically, though, the geometry of the moduli space of vacua is more complicated, in particular it will acquire many components. The twisted superpotential is a meromorphic function on the moduli space. We show that the critical points of this function determine the Bethe roots of the anisotropic spin chain, the XYZ magnet. Its XXZ limit will be mapped to the three dimensional gauge theory compactified on a circle. We thus get a satisfying picture of the elliptic, trigonometric, and rational theories corresponding to the four dimensional, three dimensional and the two dimensional theories respectively.

Our duality between the gauge theories and the quantum integrable systems can be used to enrich both subjects.

A longer version. This note is a shortened version of [19]. In [20] we give all the details covering the correspondence between vacuum structure of supersymmetric gauge theories and quantum integrable models from all perspectives, including the siring theory realization. Here we just mention that the guiding equations for the supersymmetric vacua for the two, three, and four dimensional models (compactified on the tori of appropriate dimension) can be summarized as:

$$\exp\left(\frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i}\right) = 1 \tag{1}$$

where $\tilde{W}^{eff}(\sigma)$ is the effective twisted superpotential, while σ_i are the eigenvalues of the complex scalar in the vector multiplet. It is this equation that coincides with the Bethe equation determining the exact spectrum of a quantum integrable system. In this correspondence $\tilde{W}^{eff}(\sigma)$ coincides with Yang-Yang function $Y(\lambda)$ (λ_i denoting the rapidities) counting the Bethe roots in quantum integrable systems:

$$\boxed{\begin{array}{c} Y(\lambda) \leftrightarrow \tilde{W}^{\text{eff}}(\sigma) \\ \lambda \leftrightarrow \sigma \end{array}} \tag{2}$$

We identify these quantum integrable systems in all our examples and study the consequences. In [19] the Hamiltonians of the quantum integrable system are identified with the operators of quantum multiplication in the equivariant cohomology of the hyperkähler quotients, corresponding to the Higgs branches of our gauge theories. In particular, the length L inhomogeneous $XXX_{\frac{1}{2}}$ chain (with all local spins equal to $\frac{1}{2}$) corresponds to the

[1] Another possible source of confusion is the emergence of the Bethe ansatz and the spin chains in the $\mathcal{N}=4$ supersymmetric gauge theory in four dimensions. In the work [21] and its further developments [22] the anomalous dimensions of local operators of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory are shown (to a certain loop order in perturbation theory) to be the eigenvalues of some spin chain Hamiltonian. The gauge theory is studied in the 't Hooft large N limit. In our story the gauge theory has less supersymmetry, N is finite, and the operators we consider are from the chiral ring, i.e. their conformal dimensions are not corrected quantum mechanically. Our goal is to determine their vacuum expectation values.

equivariant quantum cohomology of the cotangent bundle $T^*Gr(N,L)$ to the Grassmanian $Gr(N,L)$. This result complements nicely the construction of H. Nakajima and others of the action of the Yangians [25, 26] and quantum affine algebras on the classical cohomology and K-theory respectively of certain quiver varieties. Next, [19] applies these results to the two dimensional topological field theories. We discuss various twists of our supersymmetric gauge theories. The correlation functions of the chiral ring operators map to the equivariant intersection indices on the moduli spaces of solutions to various versions of the two dimensional vortex equations, with what is mathematically called the Higgs fields taking values in various line bundles (in the case of Hitchin equations the Higgs field is valued in the canonical line bundle). The main body of [19] has essentially shown that all known Bethe ansatz-soluble integrable systems are covered by our correspondence. However, there are more supersymmetric gauge theories which lead to the equations (1) which can be viewed as the deformations of Bethe equations. For example, a four dimensional $\mathscr{N}=2^*$ theory compactified on $\mathbf{S}^2$ with a partial twist leads to a deformation of the non-linear Schrödinger system with interesting modular properties (we devote last section of current paper to this example). Another interesting model comes from the quantum cohomology of instanton moduli spaces and the Hilbert scheme of points.

The long paper [19] and the example of the quantum cohomology of $T^*Gr(N,L)$ and its relation to the Heisenberg magnet are reviewed in detail in [20].

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THE GAUGE THEORY

Here we give a brief review of the relevant gauge theories.

Gauge theories with four supercharges

We study two dimensional $\mathscr{N}=(2,2)$ supersymmetric gauge theory with some matter. The matter fields are generally in the chiral multiplets which we denote by the letters $\mathbf{Q}$, $\tilde{\mathbf{Q}}$, and Φ (sometimes we use $\mathbf{X}$ to denote matter fields without reference to their gauge representation type), the gauge fields are in the vector multiplet $\mathbf{V}$. We also use the twisted chiral multiplets Σ, as e.g. the field strength $\Sigma = \mathscr{D}_+\bar{\mathscr{D}}_-\mathbf{V}$ is in the twisted chiral multiplet.

$$\begin{aligned}\mathbf{V} = \theta^-\bar\theta^-(A_0-A_1)+\theta^+\bar\theta^+(A_0+A_1)-\sqrt{2}\sigma\theta^-\bar\theta^+ \\ -\sqrt{2}\bar\sigma\theta^+\bar\theta^- + 2i\theta^-\theta^+(\bar\theta^-\bar\lambda_- + \bar\theta^+\bar\lambda_+)+ \\ +2i\bar\theta^+\bar\theta^-(\theta^+\lambda_+ + \theta^-\lambda_-) + 2\theta^-\theta^+\bar\theta^-\bar\theta^+ H \quad (3)\end{aligned}$$

where we use a notation H for the auxiliary field (in most textbooks it is denoted by D).

$$\mathbf{X} = X(y)+\sqrt{2}\left(\theta^+\psi_+(y)+\theta^-\psi_-(y)\right)+\theta^+\theta^-F(y) \quad (4)$$

where

$$y^{\pm} = x^{\pm} - i\theta^{\pm}\bar\theta^{\pm}\ ,$$

and the twisted chiral multiplet Σ:

$$\begin{aligned}\Sigma = \sigma(\tilde y) + i\sqrt{2}\left(\theta^+\bar\lambda_+(\tilde y) - \bar\theta^-\lambda_-(\tilde y)\right) + \\ +\sqrt{2}\theta^+\bar\theta^-\left(H(\tilde y) - iF_{01}\right) \quad (5)\end{aligned}$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1]$ is the gauge field strength, and

$$\tilde y^{\pm} = x^{\pm} \mp i\theta^{\pm}\bar\theta^{\pm}$$

[2] The IHES seminars and the theoretical physics conference dedicated to the 50th anniversary of IHES (Bures-sur-Yvette, June 2007, April 2008, June 2008); the IAS Workshop on “Gauge Theory and Representation Theory” and the IAS seminar (Princeton, November 2007, 2008); the YITP/RIMS conference “30 Years of Mathematical Methods in High Energy Physics ” in honour of 60th anniversary of Prof. T. Eguchi (Kyoto, March 2008); the London Mathematical Society lectures at Imperial College (London, April 2008); L. Landau’s 100th anniversary theoretical physics conference (Chernogolovka, June 2008); Cargese Summer Institute (Cargese, June 2008); the Sixth Simons Workshop “Strings, Geometry and the LHC” (Stony Brook, July 2008); the ENS summer institute (Paris, August 2008); the French-Japanese Scientific Forum ”Perspectives in mathematical sciences”, (Tokyo, October 2008)

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Lagrangians

The action of the corresponding two dimensional quantum field theory action has three types of terms - the D-terms, the F-terms and the twisted F-terms:

$$\begin{aligned} D: &\quad \int d^2x d^4\theta \,\mathrm{tr}\left(\Sigma\bar{\Sigma}\right) + \mathbf{K}(e^{\mathbf{V}/2}\mathbf{X}\,,\bar{\mathbf{X}}e^{\mathbf{V}/2}) \\ F: &\quad \int d^2x d\theta^+ d\theta^- \; W(\mathbf{X}) + \text{c.c.} \\ F^{\mathrm{tw}}: &\quad \int d^2x d\theta^+ d\bar{\theta}^- \; \tilde{W}(\Sigma) + \text{c.c.} \end{aligned} \tag{6}$$

Global symmetries and twisted masses

The typical $\mathscr{N} = (2,2)$ gauge theory has the matter fields $\mathbf{X}$ transforming in some linear[4] representation $\mathscr{R}$ of the gauge group G. Let us specify the decomposition of $\mathscr{R}$ onto the irreducible representations of G:

$$\mathscr{R} = \oplus_{\mathbf{i}} \mathbf{M_i} \otimes R_{\mathbf{i}} \tag{7}$$

where $R_{\mathbf{i}}$ are the irreps of G, and $\mathbf{M_i}$ are the multiplicity spaces. The group

$$H^{\max} = \times_{\mathbf{i}} U(\mathbf{M_i}) \tag{8}$$

acts on $\mathscr{R}$ and this action commutes with the gauge group action. The actual global symmetry group H of the theory may be smaller then (8) : $H \subset H^{\max}$, as it has to preserve both D and the F-terms in the action.

The theory we are interested in can be deformed by turning on the so-called twisted masses $\tilde{m}$ [27], which belong to the complexification of the Lie algebra of the maximal torus of H:

$$\tilde{m} = (\tilde{m}_{\mathbf{i}})\,,\, \tilde{m}_{\mathbf{i}} \in \mathrm{End}\,(\mathbf{M_i}) \cap H \tag{9}$$

The superspace expression for the twisted mass term is [28, 29]:

$$\mathscr{L}_{\widetilde{\mathrm{mass}}} = \int d^4\theta \; \mathrm{tr}_{\mathscr{R}} \mathbf{X}^{\dagger} \left(\sum_{\mathbf{i}} e^{\tilde{V}_{\mathbf{i}}} \otimes \mathrm{Id}_{R_{\mathbf{i}}} \right) \mathbf{X} \tag{10}$$

where

$$\tilde{V}_{\mathbf{i}} = \tilde{m}_{\mathbf{i}}\, \theta_+ \bar{\theta}_- \tag{11}$$

The twisted masses which preserve the $\mathscr{N} = 4$ supersymmetry will be denoted by μ, and the ones which break it down to $\mathscr{N} = 2$, by u.

When the twisted masses are turned on in the generic fashion, the matter fields are massive and can be integrated out. As a result, the theory becomes an effective pure $\mathscr{N} = 2$ gauge theory with an infinite number of interaction terms in the Lagrangian, with the high derivative terms suppressed by the inverse masses of the fields we integrated out. Of all these terms the F-terms, i.e. the effective superpotential, or the twisted F-terms, i.e. the effective twisted superpotential, can be computed exactly. In fact, these terms only receive one-loop contributions. Let $\tilde{\mathbf{m}}$ denote collectively the set of the twisted masses of the fields we are integrating out. We get:

$$\begin{aligned} \widetilde{W^{\mathrm{eff}}}_{\mathrm{matter}}(\sigma) = \sum_{\mathbf{b}} 2\pi i t_{\mathbf{b}} \mathrm{tr}_{\mathbf{b}} \sigma + \\ + \mathrm{tr}_{\mathscr{R}}(\sigma + \tilde{\mathbf{m}})\left(\log(\sigma + \tilde{\mathbf{m}}) - 1\right) \end{aligned} \tag{12}$$

where for each $U(1)$ factor in G we have introduced a Fayet-Illiopoulos term which together with the corresponding theta-angle combine into a complex coupling $t_{\mathbf{b}}$,

$$t_{\mathbf{b}} = \frac{\vartheta_{\mathbf{b}}}{2\pi} + i r_{\mathbf{b}}\,. \tag{13}$$

The generator of the corresponding $U(1)$ factor in G is denoted in (12) by $\mathrm{tr}_{\mathbf{b}}\sigma$. We put the subscript “matter” in (12) in order to stress the fact that it only includes the loops of the matter fields.

There are other massive fields which can be integrated out on the Coulomb branch. For example, the $\mathbf{g}/\mathbf{t}$-components of the vector multiplets (where $\mathbf{g}$ denotes Lie algebra corresponding to Lie groups G and $\mathbf{t}$ is its Cartan sub-algebra), the W-bosons and their superpartners. Their contribution to the effective twisted superpotential is rather simple:

$$\tilde{W}^{\mathrm{eff}}_{\mathrm{gauge}} = -\sum_{\alpha \in \Delta} \langle \alpha, \sigma \rangle \left[\log \langle \alpha, \sigma \rangle - 1\right] = -2\pi i \langle \rho, \sigma \rangle \tag{14}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \tag{15}$$

is half the sum of the positive roots of $\mathbf{g}$. It may appear that the expression (14) is inconsistent with the gauge invariance, however the effective interaction (14) is gauge invariant. The total effective twisted superpotential is, therefore:

$$\tilde{W}^{\mathrm{eff}}(\sigma) = \tilde{W}^{\mathrm{eff}}_{\mathrm{matter}}(\sigma) + \tilde{W}^{\mathrm{eff}}_{\mathrm{gauge}}(\sigma) \tag{16}$$

Superpotential deformations and twisted masses

The supersymmetric field theories also have the superpotential deformations, which correspond to the F-terms in (6). The superpotential W has to be a holomorphic gauge invariant function of the chiral fields, such as $\Phi, Q, \tilde{Q}$. It may be not invariant under the maximal symmetry group $H^{\max}$, thus breaking it to a

[4] In [19] we also discuss the generalization where $\mathbf{X}$ takes values in some non-linear space with the G-action

subgroup H or completely. For example, the so-called *complex mass* of the fundamental and anti-fundamental fields [30] comes from the superpotential $W_{\text{complex mass}} = \sum_{a,b} m_a^b \tilde{Q}_b Q^a$, which breaks the $U(n_{\mathbf{f}}) \times U(n_{\bar{\mathbf{f}}})$ group down to $U(1)^{\min(n_{\mathbf{f}}, n_{\bar{\mathbf{f}}})}$.

In all cases discussed in this paper, in spacetime dimensions two, three and four, one can consider more sophisticated superpotentials, involving the fundamental, anti-fundamental, and adjoint chiral fields:

$$W_{\tilde{Q}\Phi Q} = \sum_{a,b} \tilde{Q}^a m_a^b(\Phi) Q_b = \sum_{a,b;s} m_{a;s}^b \tilde{Q}^a \Phi^{2s} Q_b \qquad (17)$$

The case of most interest for us, that of the two dimensional ultraviolet finite theories corresponds to $n_{\mathbf{f}} = n_{\bar{\mathbf{f}}} = L$. In this case we will see later that equations describing supersymmetric vacua are linked to known quantum integrable lattice models.

Examples

There are two classes of examples: *a*.) the asymptotically free theories and *b*.) the asymptotically conformal theories. The *a*.) examples include the gauge theories which look at low energy as the $\mathscr{N} = 2$ sigma models with various Kähler target spaces: the complex projective space $\mathbf{CP}^{L-1}$, the Grassmanian $\mathrm{Gr}(N,L)$, or, more generally, the (partial) flag variety $F(n_1, n_2, \ldots, n_{\mathbf{r}}, n_{\mathbf{r}+1} \equiv L)$. The *b*.) examples can also be identified at the low energy level with the sigma models. These sigma models typically have the hyperkähler target spaces, such as the cotangent bundles to the Kähler manifolds from the *a*.) list. The *b*.) examples turn out to include, via (1), essentially all known quantum integrable models of statistical physics.

Of course, by taking an appropriate scaling limit one can get the *a*.) models from the *b*.) models. For example, the Grassmanian model (which is so extensively studied in [31]) is a limit of the $T^*\mathrm{Gr}(N,L)$ model in the limit where the twisted mass u corresponding to the rotations of the cotangent direction is sent to infinity, with the complexified Kähler class adjusted in such a way, that the effective mass scale:

$$\Lambda_{\mathrm{Gr}} = u e^{\frac{2\pi i t}{L}}$$

remains finite. This corresponds to a non-Hermitian deformation of the Heisenberg magnet which is dual, via (1), to the original $T^*\mathrm{Gr}(N,L)$ theory.

The reason why the ultraviolet finiteness is so special in the relation to the quantum integrability has to do with the S-matrix nature of the Bethe equations which we identify with the vacuum equation (1).

In this note we consider the $G = U(N)$ gauge group only. Here we present the effective twisted superpotential (16) for the main example of the *b*.) class. There are many more examples presented in [19].

Two dimensions

One can start with the so-called $\mathscr{N} = 2^*$ theory. It has $\mathscr{R} = \mathbf{g} \otimes \mathbf{C}$, i.e. the adjoint chiral multiplet $\blacksquare$. In the absence of the twisted mass term this is the $\mathscr{N} = 4$ theory, the dimensional reduction of the pure $\mathscr{N} = 2$ super-Yang-Mills from four dimensions. This theory has a global $U(1)$ symmetry, which rotates the adjoint chiral multiplet, e.g. $\blacksquare \mapsto e^{i\varphi}\blacksquare$. We can turn on the corresponding twisted mass $\tilde{m} = iu$ which breaks $N = 4$ to $N = 2$ (the factor of i is introduced for the later convenience). The effective twisted superpotential for $G = U(N)$ is:

$$\begin{aligned}\tilde{W}^{\mathrm{eff}}(\sigma) = & \sum_{i,j=1}^{N} (\sigma_i - \sigma_j + iu)\left(\log(\sigma_i - \sigma_j + iu) - 1\right) \\ & - 2\pi i \sum_{i=1}^{N} \left(t + i - \frac{1}{2}(N+1)\right)\sigma_i \qquad (18)\end{aligned}$$

A more interesting theory is obtained by taking

$$\mathscr{R} = V \otimes V^* \otimes \mathscr{L} \oplus V \otimes \mathscr{F} \oplus V^* \otimes \tilde{\mathscr{F}} \ .$$

which corresponds to the theory with the $H^{\max} = U(L) \times U(L) \times U(1)$ global symmetry group. Here $V = \mathbf{C}^N$ is the N-dimensional fundamental representation of G, $\mathscr{F} \approx \mathbf{C}^L$, $\tilde{\mathscr{F}} \approx \mathbf{C}^L$ are the L-dimensional fundamental representations of the first and the second $U(L)$ factors in the flavour group, and $\mathscr{L}$ is the standard one-dimensional representation of the global group $U(1)$. In simple terms, this theory has the matter content of the four dimensional $N_c = N$, $N_f = L$, $\mathscr{N} = 2$ gauge theory with fundamental hypermultiplets, however, the supersymmetry is half that of the four dimensional theory. This theory has $2L+1$ twisted mass parameters (we skip tildes from now on): $(m_a^{\mathrm{f}}, m_a^{\bar{\mathrm{f}}})_{a=1}^{L}, m^{\mathrm{adj}} = -iu$. Upon integrating out the matter fields and the W-bosons we get the theory of the Abelian vector multiplet with the effective twisted superpotential:

$$\begin{aligned}\tilde{W}^{\mathrm{eff}}_{\tilde{Q}\Phi Q}(\sigma) = & \sum_{i=1}^{N}\sum_{a=1}^{L}\Big[\left(\sigma_i + m_a^{\mathrm{f}}\right)\left(\log\left(\sigma_i + m_a^{\mathrm{f}}\right) - 1\right) \\ & + \left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right)\left(\log\left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right) - 1\right)\Big] + \\ & + \sum_{i,j=1}^{N}\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right)\left(\log\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right) - 1\right) - \\ & - 2\pi i \sum_{i=1}^{N}\left(t + i - \frac{1}{2}(N+1)\right)\sigma_i \qquad (19)\end{aligned}$$

The generic twisted masses are incompatible with any tree level superpotential. However, for the special choice of the twisted masses one can turn on the tree level superpotential. Its variation does not change the effective twisted superpotential (19) though. We shall discuss this point later.

Three dimensions

Consider now the theory on $\mathbf{R}^2 \times \mathbf{S}^1$. It suffices to make all the fields depend on an extra coordinate $x^2 = y$, $y \sim y + 2\pi$. Since the translations in y are the global symmetry of the theory we can turn on the corresponding twisted mass $\tilde{m}$[5]. This is equivalent to promoting the real part of the complex scalar in the vector multiplet to the covariant derivative:

$$\begin{aligned} \sigma(t,x) &\longrightarrow \frac{1}{R}\partial_y + \sigma(t,x,y)\ , \\ \bar{\sigma}(t,x) &\longrightarrow -\frac{1}{R}\partial_y + \bar{\sigma}(t,x,y) \end{aligned} \tag{20}$$

where R is the radius of the circle $\mathbf{S}^1$. In other words,

$$\sigma = \frac{1}{R}A_y + \sigma_{\mathbf{R}} \tag{21}$$

where A_y is the gauge field component (the y coordinate being dimensionless the A_y field is dimensionless too, while σ has a dimension of mass). The twisted mass corresponding to the translations is $\tilde{m} = \frac{i}{R}$. Thus, the Kaluza-Klein modes with momentum n, $n \in \mathbf{Z}$, have the corresponding twisted mass

$$\tilde{m}_n = \frac{in}{R} \tag{22}$$

To compute the effective twisted superpotential, it suffices to enumerate the Kaluza-Klein modes and sum up their contributions. One needs to use a kind of zeta-regularization, which can be justified, e.g. by topological field theory methods [24].

For definiteness let us consider the contribution of a matter field in the representation $\mathscr{R}$ of the gauge group. Let $\tilde{\mathbf{m}}$ denote the ordinary two dimensional twisted mass, corresponding to the centralizer of G in $\mathscr{R}$ which preserves other couplings of the theory, such as the superpotential. We assume $\tilde{\mathbf{m}}$ sufficiently generic so that all the modes of the corresponding matter multiplet are massive.

[5] The space of fields is of course acted on by $\mathrm{Diff}(S^1)$, but the Lagrangian is invariant only under S^1, the translations.

The sum over the Kaluza-Klein modes gives:

$$\begin{aligned} &\tilde{W}^{\mathrm{eff}}_{\mathrm{matter}}(\sigma) = \\ &\mathrm{tr}_{\mathscr{R}}\left[\sum_{n\in\mathbf{Z}}\left(\sigma + \tilde{\mathbf{m}} + \frac{in}{R}\right)\left(\log\left(\sigma + \tilde{\mathbf{m}} + \frac{in}{R}\right) - 1\right)\right] \sim \\ &\sim \frac{1}{2\pi R}\mathrm{tr}_{\mathscr{R}}\left[\mathrm{Li}_2\left(e^{-2\pi R(\sigma+\tilde{\mathbf{m}})}\right)\right] \end{aligned} \tag{23}$$

In addition to the matter-induced twisted superpotential we also have a contribution of the W-bosons:

$$\begin{aligned} \tilde{W}^{\mathrm{eff}}_{\mathrm{gauge}} &= -\mathrm{tr}_{\mathbf{g}/\mathbf{t}}\left[\frac{1}{2\pi R}\mathrm{Li}_2\left(e^{-2\pi R\sigma}\right)\right] = \\ &= \frac{\pi R}{2}\mathrm{tr}_{\mathrm{adj}}\left(\sigma^2\right) + 2\pi i\langle\rho,\sigma\rangle \end{aligned} \tag{24}$$

where we used:

$$\mathrm{Li}_2(e^{-x}) + \mathrm{Li}_2(e^{x}) = \frac{\pi^2}{3} - i\pi x - \frac{x^2}{2} \tag{25}$$

and dropped an irrelevant constant. The quadratic term in (24) corresponds to the anomaly-induced Chern-Simons interaction [32, 33] in the three dimensional theory.

Four dimensions

We can lift the theory to the $\mathscr{N} = 1$ supersymmetric Yang-Mills theory (with matter), compactified on a two-torus $\mathbf{T}^2$. Again, we can view the lift to four dimensions as the two dimensional theory with the infinite number of fields, which depend on the two additional coordinates (y,z), with $y \sim y + 2\pi$, $z \sim z + 2\pi$. The theory is regularized by the twisted masses corresponding to the translations along $\mathbf{T}^2$. We choose one of the masses to be $\frac{i}{R}$, then the other is $\frac{i\tau}{R}$. Here τ is the complex modulus of $\mathbf{T}^2$. The normalized holomorphic coordinate on $\mathbf{T}^2$ is given by: $w = \frac{1}{2\pi}(y + \tau z)$. The gauge theory is sensitive to the metric on the torus and a two-form, the so-called B-field, via the coupling

$$\int_{\mathbf{R}^2\times\mathbf{T}^2} B \wedge \mathrm{tr}F\ . \tag{26}$$

Similarly to the three dimensional lift of the previous section the field σ gets promoted to the covariant derivative operator ($\tau_2 = \mathrm{Im}\tau$):

$$\begin{aligned} \sigma(t,x) &\to \frac{\tau_2}{i\pi R}\bar{\partial} + \sigma(t,x,y,z), \\ \bar{\sigma}(t,x) &\to \frac{\tau_2}{i\pi R}\partial + \bar{\sigma}(t,x,y,z) \end{aligned} \tag{27}$$

where

$$\bar{\partial} = \frac{i\pi}{\tau_2}\left(\partial_z - \tau\partial_y\right)$$

The invariance under the large gauge transformations now translates to the double-periodicity of the twisted superpotential:

$$\sigma \to \sigma + \frac{i}{R}(m + n\tau)\ , m, n \in \mathbf{Z} \tag{28}$$

The effective twisted superpotential is given by ($\mathrm{q} = \exp 2\pi i \tau$):

$$\begin{aligned}\tilde{W}^{\mathrm{eff}} = \frac{\pi R}{2}\mathrm{tr}_{\mathscr{R}}(\sigma + \tilde{\mathbf{m}})^2 + \frac{\pi i \tau}{6}\mathrm{tr}_{\mathscr{R}}(\sigma) + \frac{1}{2\pi R}\times \\ \sum_{n=1}^{\infty}\mathrm{tr}_{\mathscr{R}}\left[\mathrm{Li}_2\left(\mathrm{q}^{n-1}e^{-2\pi R(\sigma+\tilde{\mathbf{m}})}\right) - \mathrm{Li}_2\left(\mathrm{q}^{n}e^{2\pi R(\sigma+\tilde{\mathbf{m}})}\right)\right] \\ = \frac{\pi R}{2}\mathrm{tr}_{\mathscr{R}}(\sigma + \tilde{\mathbf{m}})^2 + \frac{\pi i \tau}{6}\mathrm{tr}_{\mathscr{R}}(\sigma) + \\ + \frac{1}{2\pi R}\sum_{n \in \mathbf{Z}_{\neq 0}}\frac{\mathrm{tr}_{\mathscr{R}}\left[e^{2\pi R n(\sigma+\tilde{\mathbf{m}})}\right]}{n^2(1-\mathrm{q}^n)}\end{aligned} \tag{29}$$

plus linear terms.

Supersymmetric vacua of N=2 theories

The only local gauge invariant of the Abelian gauge field in two dimensions is the field strength F_{01} which is subject to the only global constraint:

$$\frac{1}{2\pi i}\int_{\Sigma} F^i = m^i \in \mathbf{Z} \tag{30}$$

i.e. the integrality of the magnetic flux. In addition, the global invariants of the $\mathbf{T}$-valued gauge field include the holonomies, which are irrelevant for our discussion at the moment.

In order to minimize the potential energy and find the vacua of the theory we promote F_{01}^{α} to the independent fields, while adding at the same time the term

$$\sum_{i=1}^{r} n_i \int_{\Sigma} F^i \tag{31}$$

to the action (cf. [34, 35, 36]). Following [37], the shift (31) is equivalent to the shift

$$\tilde{W}^{\mathrm{eff}}(\sigma) \longrightarrow \tilde{W}^{\mathrm{eff}}_{\vec{n}}(\sigma) = \tilde{W}^{\mathrm{eff}}(\sigma) - 2\pi i \sum_{i=1}^{r} n_i \sigma^i \tag{32}$$

where now $D \pm iF$ are two independent auxiliary fields, which can be integrated out. Thus the target space of the effective sigma model becomes, *a priori*, disconnected, with $\vec{n}$ labeling the connected components. In fact, the actual connected components are labeled by the equivalence classes of $\vec{n}$ up to the action of the monodromy group (the effective superpotential is not a univalent function of σ). The potential on the component, labelled by $\vec{n}$ is given by (note that unlike the standard expressions involving "$\ldots \min_n (x + 2\pi n)^2$, which follows from the pair creation in the background electric field induced by the theta angle... " it is consistent with supersymmetry and holomorphy):

$$U_{\vec{n}} = \frac{1}{2}\mathrm{g}^{ij}\left(-2\pi i n_i + \frac{\partial \tilde{W}^{\mathrm{eff}}}{\partial \sigma^i}\right)\left(+2\pi i n_j + \frac{\partial \tilde{\bar{W}}^{\mathrm{eff}}}{\partial \bar{\sigma}^j}\right) \tag{33}$$

The minima of the effective potential (33) are thus the solutions of the equations:

$$\frac{1}{2\pi i}\frac{\partial \tilde{W}^{\mathrm{eff}}(\sigma)}{\partial \sigma^i} = n_i \tag{34}$$

This equation is derived under very general conditions. Everything is hidden in $\tilde{W}^{\mathrm{eff}}$. The n_i dependence in (34) can be eliminated by exponentiating both sides:

$$\boxed{\exp\left(\frac{\partial \tilde{W}^{\mathrm{eff}}(\sigma)}{\partial \sigma^i}\right) = 1} \tag{35}$$

Examples of the vacuum equations

Old examples: asymptotically free theories

Asymptotically free theories are certain limits of asymptotically conformal theories. Since our main examples are asymptotically conformal for completeness we give couple of examples of asymptotically free theories first.

<u>$\mathbf{CP}^{L-1}$ model</u>. $G = U(1), \mathscr{R} = R_{+1} \otimes \mathbf{C}^L$, where R_{+1} is a one-dimensional charge $+1$ representation of $U(1)$. From twisted effective suprpotential of this model we immediately derive:

$$\prod_{\alpha=1}^{L}(\sigma + \tilde{m}_{\alpha}) = q \equiv e^{2\pi i t} \tag{36}$$

which implies that the model has L isolated vacua, and the theory at each vacuum is massive, for the generic values of the twisted masses $\tilde{m}_{\alpha}$. For vanishing twisted masses the equation (36) simplifies to $\sigma^L = q$ which is the famous quantum cohomology ring of $\mathbf{CP}^{L-1}$. For the generic twisted masses the equation (36) describes the $U(L)$-equivariant quantum cohomology $H^{*}_{U(L)}(\mathbf{CP}^{L-1})$ ring.

The next example is that of the

Grassmanian $Gr(N,L)$ model. $G = U(N)$ and $\mathscr{R} = \mathbf{C}^N \otimes \mathbf{C}^L$. Using the effective twisted superpotential of this model we derive:

$$\prod_{\alpha=1}^{L} (\sigma_l + \tilde{m}_\alpha) = (-1)^{N+1} e^{2\pi i t} , \ l = 1, \ldots, N \tag{37}$$

We should supplement the equations (37) with the condition that $\sigma_l \neq \sigma_m$ for $l \neq m$ and identify the solutions which differ by the permutations of σ_l's. In other words, the equations (37) should be viewed as equations on the elementary symmetric functions

$$c_l = \sum_{i_1 < \ldots < i_l} \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_l} \tag{38}$$

which can be compactly written using the gauge invariant order parameter $\mathbf{Q}(x)$,

$$\mathbf{Q}(x) \equiv \det(x - \sigma) = \prod_{i=1}^{N} (x - \sigma_i) = x^N + \sum_{i=1}^{N} (-1)^i c_i x^{N-i} , \tag{39}$$

which we shall call the Baxter-Chern (BC) order parameter, as:

$$\prod_{\alpha=1}^{L} (x + \tilde{m}_\alpha) + (-1)^N q = t(x) \mathbf{Q}(x) \tag{40}$$

for some polynomial $t(x)$ of degree $L - N$,

$$t(x) = x^{L-N} + \sum_{j=1}^{L-N} t_j x^{L-N-j} .$$

This polynomial is uniquely fixed in terms of c_i's from the equation (40) by expanding both sides at $x = \infty$ and equating the coefficients of x^{L-N-j}, $j = 1, \ldots, L-N$. In the classical limit $q \to 0$ the polynomial $\mathbf{Q}(x)$ is essentially the $U(L)$-equivariant Chern polynomial of the tautological rank N bundle E over the Grassmanian $\mathrm{Gr}(N,L)$, while $t(x)$ is the $U(L)$-equivariant Chern polynomial of the tautological dual bundle $E^\perp$ of rank $L-N$. The relation (40) then reads simply as the consequence of the exactness of the sequence:

$$0 \longrightarrow E \longrightarrow F \approx \mathbf{C}^L \longrightarrow E^\perp \longrightarrow 0$$

$\mathscr{N} = 2^*$ theory. The example of the pure $\mathscr{N} = 4$ theory broken down to $\mathscr{N} = 2$ by the twisted mass term for the adjoint chiral multiplet is the first example where the supersymmetry is broken, for $N > 1$. Here $G = U(N)$, $SU(N)$, $SO(N)$, $Sp(N)$ and $\mathscr{R} = \mathbf{g} \otimes \mathbf{C}$, i.e. the adjoint representation. Using (18) we derive:

$$\prod_{j=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j - m} = -q , \tag{41}$$

which can be neatly rewritten using our $\mathbf{Q}$-operator (39) again:

$$\mathbf{Q}(x+m) + q\, \mathbf{Q}(x-m) = (1+q)\, \mathbf{Q}(x) \tag{42}$$

It is easy to see that this equation has no solutions for σ_i's for $N > 2$, or for $N = 1, q \neq 1$ and has a valley of solutions for $N = 1$, $q = 1$.

Hitchin theory. The model studied in [1, 3, 4] corresponds to the $\mathscr{N} = 2^*$ theory with the tree level *twisted* superpotential[6]:

$$\tilde{W}(\sigma) = \frac{\lambda}{2} \mathrm{tr}\, \sigma^2 , \tag{43}$$

which corresponds to the two-observable representing the Kähler form on the Hitchin's moduli space $\mathscr{M}_H$. This leads to the change in the right hand side of (41):

$$\prod_{j=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j - m} = \exp 2\pi i \lambda\, \sigma_i , \tag{44}$$

and one now gets solutions for σ_i's for all N. The topological twist of this theory, introduced in [1] and was studied in detail in [3, 4].

New examples: asymptotically conformal theories

Our main example will be the $U(N)$ gauge theory with L fundamental chiral multiplets $\mathbf{Q}_a$, L anti-fundamental chiral multiplets $\tilde{\mathbf{Q}}^a$, and one adjoint chiral multiplet Φ. This matter content corresponds to the gauge theory with extended supersymmetry, $\mathscr{N} = 4$, which the dimensional reduction of the four dimensional $\mathscr{N} = 2$ theory. The adjoint Φ is a part of the vector multiplet in four dimensions, while the chiral fundamental and anti-fundamentals combine into the four dimensional hypermultiplet in the fundamental representation. We are dealing, therefore, with the matter content of the four dimensional $\mathscr{N} = 2$ theory with $N_c = N$, $N_f = L$. If the superpotential $\sum_a \tilde{Q}^a \Phi Q_a$ is added, then the theory *does* have the four dimensional $\mathscr{N} = 2$ supersymmetry.

Since the gauge group has a center $U(1)$ one can turn on the Fayet-Illiopoulos term, and the theta angle as we already explained, which we combine into a complexified coupling $\vartheta \mapsto t = \frac{\vartheta}{2\pi} + ir$.

First, we consider the theory with general twisted masses for the chiral fundamentals, anti-fundamentals, and the adjoint field (which is compatible only with the zero superpotential). We then turn on the superpotential and discuss the consequences.

[6] In most of the discussion we have the tree level *superpotential*, rather then the tree level *twisted superpotential* turned on.

Two dimensions. Using (34) with (19) we arrive at the equations for vacua (we shift t by $L/2$ to avoid extra phases in the right hand side):

$$\prod_{a=1}^{L} \frac{\sigma_i + m_a^{\mathrm{f}}}{\sigma_i - m_a^{\bar{\mathrm{f}}}} = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\sigma_i - \sigma_j - m^{\mathrm{adj}}}{\sigma_i - \sigma_j + m^{\mathrm{adj}}} \qquad (45)$$

The equation (45) is written in terms of the eigenvalues σ_i of the complex scalar σ. The equations have solutions related by permuting σ_i's. These solutions are physically equivalent. It is better to formulate (45) directly in the gauge invariant terms. This is done, similar to Grassmanian case above, with the help of the BC order parameter (39). The equation (45) is equivalent to:

$$a(x)\,\mathbf{Q}(x+m^{\mathrm{adj}}) + e^{2\pi i t}\, d(x)\mathbf{Q}(x-m^{\mathrm{adj}}) = t(x)\mathbf{Q}(x) \qquad (46)$$

where:

$$a(x) = \prod_{a=1}^{L}(x+m_a^{\mathrm{f}}) \ , \ d(x) = \prod_{a=1}^{L}(x-m_a^{\bar{\mathrm{f}}}) \qquad (47)$$

and $t(x)$ is an unknown polynomial of degree L.

Three dimensions. If we take the analogous theory in three dimensions, compactified on a radius R circle, the resulting vacuum equations would look like:

$$\prod_{a=1}^{L} \frac{\sinh\left(\pi R\left(\sigma_i + m_a^{\mathrm{f}}\right)\right)}{\sinh\left(\pi R\left(\sigma_i - m_a^{\bar{\mathrm{f}}}\right)\right)} = \\ = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\sinh\left(\pi R\left(\sigma_i - \sigma_j - m^{\mathrm{adj}}\right)\right)}{\sinh\left(\pi R\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right)\right)} \qquad (48)$$

Notice the invariance of the eqs. (48) under the transformations:

$$\sigma_i \longrightarrow \sigma_i + \frac{i n_i}{R} \, , \, n_i \in \mathbf{Z} \qquad (49)$$

and the permutations of σ_i's. This invariance is the affine Weyl group symmetry, the residual gauge invariance, whose origin is the gauge transformations of the form:

$$g(y) = \mathrm{diag}\left(e^{i n_1 y}, \ldots, e^{i n_N y}\right) \, .$$

The equations (48) can be also analyzed in the gauge invariant fashion using the BC operator. The order parameters of the three dimensional theory compactified on the circle $\mathbf{S}^1$ are contained in the trigonometric polynomial (cf. [24]):

$$\mathbf{Q}(x) = 2^N e^{\pi R N \hat{y}} \prod_{i=1}^{N} \sinh\left(\pi R\left(\hat{y} - \sigma_i\right)\right) = \\ = x^N + u_1 x^{N-1} + \ldots + u_N \, , \qquad (50)$$

where

$$x = \exp\left(2\pi R \hat{y}\right) \qquad (51)$$

The equations (48) are equivalent to the difference equation:

$$a(x)\mathbf{Q}(x\hat{q}) + q\, d(x)\mathbf{Q}(x\hat{q}^{-1}) = t(x)\mathbf{Q}(x) \qquad (52)$$

where $q = e^{2\pi i t}$,

$$\hat{q} = e^{2\pi R m^{\mathrm{adj}}} \, , \\ a(x) = \prod_{a=1}^{L}\left(x e^{\pi R m_a^{\mathrm{f}}} - e^{-\pi R m_a^{\mathrm{f}}}\right) , \\ d(x) = \prod_{a=1}^{L}\left(x e^{-\pi R m_a^{\bar{\mathrm{f}}}} - e^{+\pi R m_a^{\bar{\mathrm{f}}}}\right) \qquad (53)$$

and $t(x)$ is a polynomial to be determined.

In the limit $R \to 0$ with all other parameters kept finite we recover the two dimensional story.

Four dimensions. The four dimensional gauge theory with the similar field content, compactified on a two-torus with the modular parameter τ, will lead to the elliptic generalization of (48):

$$\prod_{a=1}^{L} \frac{\Theta_1\left(\pi R\left(\sigma_i + m_a^{\mathrm{f}}\right)\right)}{\Theta_1\left(\pi R\left(\sigma_i - m_a^{\bar{\mathrm{f}}}\right)\right)} = \\ = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\Theta_1\left(\pi R\left(\sigma_i - \sigma_j - m^{\mathrm{adj}}\right)\right)}{\Theta_1\left(\pi R\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right)\right)} \qquad (54)$$

where (in this section q denotes $\exp(2\pi i \tau)$):

$$\Theta_1(\xi) = -i \mathrm{q}^{\frac{1}{8}} \left(e^{\xi} - e^{-\xi}\right) \times \\ \times \prod_{m=1}^{\infty} \left(1-\mathrm{q}^m\right)\left(1-\mathrm{q}^m e^{2\xi}\right)\left(1-\mathrm{q}^m e^{-2\xi}\right) \qquad (55)$$

The gauge invariance of the equations (54) is more subtle then that of its three and two dimensional counterparts.We have the gauge transformations of the form:

$$g(y,z) = \mathrm{diag}\left(e^{i n_1 y - i m_1 z}, \ldots, e^{i n_N y - i m_N z}\right) \, , \; n_i, m_i \in \mathbf{Z} \qquad (56)$$

which act on σ as follows:

$$\sigma_i \mapsto \sigma_i + \frac{i}{R}\left(n_i + m_i \tau\right) \qquad (57)$$

The shifts by n_i's are clearly a symmetry of (54). The shifts by m_i's are more subtle. It turns out that to maintain the invariance of (54) under these shifts one has to assume that

$$\sum_a \left(m_a^{\mathrm{f}} + m_a^{\bar{\mathrm{f}}}\right) = -N m^{\mathrm{adj}}$$

and that t transforms under the $U(1)$ subgroup of the $U(N)$ gauge transformations. The physics of this phenomenon is rather deep, as it involves the chiral anomalies of the charged fermions in four dimensions.

SPIN CHAINS AND BETHE ANSATZ

In this section we give a swift review of the integrable spin chains at the example of the XXX spin chain for $SU(2)$. We also briefly mention other models like XXZ, XYZ, spin chains with other groups, various boundary conditions, various limits, such as the one-dimensional Bose gaz, the one-dimensional Hubbard model, etc. The so-called Yang-Yang (YY) function $Y(\lambda)$ plays the central rôle in our discussion. Its critical points are the solutions of Bethe equations. These equations determine the spectrum of integrable hamiltonians. That the equations determining the spectrum have a potential is a highly non-trivial consequence of the rich algebraic structure behind these systems. It is also the cornerstone of our correspondence with the gauge theories.

XXX spin chain

The Heisenberg spin chain, also known as the $SU(2)$ XXX spin chain, is defined on the one dimensional length L lattice. At each lattice point one has the spin $s=\frac{1}{2}$ representation of $SU(2)$, and the Hilbert space of the system is the tensor product $\mathscr{H}_L = \mathbf{C}^2\otimes\mathbf{C}^2\otimes\ldots\otimes\mathbf{C}^2$. The Hamiltonian $H_{\rm Heis}$ acts in $\mathscr{H}_L$. It is written in terms of generators $\vec{S}_a=\frac{i}{2}\vec{\sigma}_a$ where a denotes the position on the lattice, the of spin $s=\frac{1}{2}$ representation of $SU(2)$ and has the nearest-neighbor interaction form:

$$H_{\rm Heis}=J\sum_{a=1}^{L}(S_a^xS_{a+1}^x+S_a^yS_{a+1}^y+S_a^zS_{a+1}^z) \qquad (58)$$

The boundary conditions are quasi-periodic:

$$\vec{S_{L+1}}=e^{\frac{i}{2}\vartheta\sigma_3}\vec{S}_1e^{-\frac{i}{2}\vartheta\sigma_3}\ . \qquad (59)$$

In other words we identify $\mathscr{H}_L$ with the subspace $\mathscr{H}_L^{\vartheta}\subset\left(\mathbf{C}^2\right)^{\otimes\infty}$, characterized by (59). One can also consider the spin chains defined on an open interval. For the ferromagnet $J>0$ and for the anti-ferromagnet - $J<0$.

The total spin, $\vec{\mathbf{S}}=\sum_{a=1}^{L}\vec{S}_a$ commutes with $H_{\rm Heis}$ for $\vartheta=0$. The spin projection on the third axis, $\mathbf{S}^z$, is a conserved quantity for any ϑ. The corresponding subspace of the Hilbert space, $\mathscr{H}_L^N\subset\mathscr{H}_L$, where $\mathbf{S}^z=N-\frac{1}{2}L$, is sometimes called the N-particle sector.

We study the N-particle eigenstates of $H_{\rm Heis}$. The states in $\mathscr{H}_L^N$ are the linear combinations of the states with N spins up and $L-N$ spins down. Clearly, the maximal number of spins up or down is L, so $|\mathbf{S}^z|\le\frac{L}{2}$, and $N\le L$. The N-particle state $|\Psi\rangle$ can be expanded as:

$$|\Psi\rangle=\sum_{1\le x_1<\ldots<x_N\le L}\Psi(x)\,|x_1,\ldots,x_N\rangle \qquad (60)$$

with $|x_1,\ldots,x_N\rangle$ denoting the state in above tensor product with spins up at the positions $x_1,\ldots,x_N$: $|x_1,\ldots,x_N\rangle=S_{x_1}^+\ldots S_{x_N}^+\Omega$, where $\Omega=|\downarrow\downarrow\ldots\downarrow\rangle$ is the (pseudo)vacuum, the state with all spins down. It is annihilated by all operators S_x^-, $S_x^-\,\Omega=0$. The total number of the N-particle eigenstates of the Hamiltonian $H_{\rm Heis}$ is $\binom{L}{N}$, as they can be enumerated by the appropriate functions $\Psi(x)$.

The coordinate Bethe ansatz

In 1931 H. Bethe parameterized [38] these functions by the N quasimomentum variables $p=(p_1,\ldots,p_N)$, subject to the further equations which we write momentarily. The ansatz, known as *Bethe ansatz*, reads as follows: let

$$\Psi_p(x_1,\ldots x_N)=$$
$$\sum_{w\in S_N}(-1)^wA(p_{w(1)},..,p_{w(N)})\exp\left(\sum_{j=1}^{N}ip_{w(j)}x_j\right)\ , \qquad (61)$$

then the eigenstate of $H_{\rm Heis}$ is given by $|\Psi_p>=\sum_{1\le x_1<\ldots<x_N\le L}\Psi_p(x)|x_1,\ldots,x_N\rangle$. The Bethe ansatz expresses the coefficients $A_p(x)$ in terms of the two body S-matrix $\Sigma(p_1,p_2)$:

$$A(p_1,\ldots,p_N)=\prod_{1\le j\le k\le N}\Sigma(p_j,p_k),$$
$$\Sigma(p_j,p_k)=1-2e^{ip_k}+e^{i(p_j+p_k)}.$$

It is more convenient to use the new variables λ_j instead of p_j:

$$e^{ip_j}=\frac{\lambda_j+\frac{i}{2}}{\lambda_j-\frac{i}{2}}\ .$$

In this notation $\Psi_\lambda(x)$ of (60) is an eigenstate of the $H_{\rm Heis}$ if and only if $(\lambda_1,\ldots,\lambda_N)$ satisfy the Bethe equation:

$$\left(\frac{\lambda_j+\frac{i}{2}}{\lambda_j-\frac{i}{2}}\right)^L=e^{i\vartheta}\prod_{k\ne j}\frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i} \qquad (62)$$

which guarantees that (61) obeys the twisted boundary conditions (59). The energy of the state (60) is $H_{\rm Heis}\Psi_p=E_p\Psi_p$, $E_p=J\left(L-2N+2\sum_{i=1}^{N}\cos(p_i)\right)$.

A similar construction works for an arbitrary spin, when $\vec{\mathbf{S}}_a$ is in the spin s_a representation of $SU(2)$ at every site of a chain. In addition, the spin sites can be, in some sense, displaced from the symmetric round-the-clock configuration, so that one gets L additional parameters $\nu_1,\ldots,\nu_L$. This model is sometimes called the inhomogeneous XXX_s magnet. The corresponding Bethe equations have the form:

$$\prod_{a=1}^{L} \frac{\lambda_j - \nu_a + is_a}{\lambda_j - \nu_a - is_a} = e^{i\vartheta} \prod_{k\neq j} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \tag{63}$$

The Hamiltonian for the general local spins is given by a polynomial in the neighboring spins, which is more complicated then (58), see [19] for details.

The analytic Bethe Ansatz

There is yet another interpretation of the Bethe equations (63), due to [39, 40, 41, 42, 43, 44]. It can be interpreted simply as the condition for the polynomial function

$$\mathbf{Q}(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_i) \tag{64}$$

to solve Baxter's equation

$$a(\lambda)\mathbf{Q}(\lambda + i) + e^{i\vartheta} d(\lambda)\mathbf{Q}(\lambda - i) = t(\lambda)\,\mathbf{Q}(\lambda) \tag{65}$$

with the given polynomials:

$$a(\lambda) = \prod_{a=1}^{L} (\lambda - \nu_a - is_a u)\,, \quad d(\lambda) = \prod_{a=1}^{L} (\lambda - \nu_a + is_a u) \tag{66}$$

and some unknown degree L polynomial $t(\lambda)$.

Indeed, let us define $t(\lambda)$ as the ratio of the left hand side of (65) and $\mathbf{Q}(\lambda)$. The absence of poles of $t(\lambda)$ at the zeroes of $\mathbf{Q}(\lambda)$, i.e. at $\lambda = \lambda_j$, $j = 1, \dots, N$ is equivalent to (63).

The polynomial $t(\lambda)$ gives the eigenvalues of the twisted transfer matrix

$$T_{\vartheta}(\lambda) = A(\lambda) + e^{i\vartheta} D(\lambda) \tag{67}$$

which a central object in *algebraic Bethe Ansatz* where it is a trace of monodromy matrix, see [19] for details.

Yang-Yang function

The highly surprising property of the equations (62),(63) is that they have a potential. If we rewrite (63) as $e^{2\pi i \varpi_j(\lambda)} = 1$, then the following one-form:

$$\varpi = \sum_{j=1}^{N} \varpi_j(\lambda) \mathrm{d}\lambda_j \tag{68}$$

is closed, $d\varpi = 0$ and $\varpi = dY$

$$\begin{aligned} Y(\lambda) = &\sum_{a=1}^{L} \frac{s_a}{\pi} \sum_{j=1}^{N} \hat{x}\left(\frac{\lambda_j - \nu_a}{s_a}\right) + \\ &+ \frac{1}{\pi} \sum_{j,k=1}^{N} \hat{x}(\lambda_j - \lambda_k) + \sum_{j=1}^{N} \lambda_j \left(n_j + \frac{\vartheta}{2\pi}\right) \end{aligned} \tag{69}$$

where the integers n_j label various branches of the logarithms, and the function $\hat{x}(\lambda)$ is given by:

$$\hat{x}(\lambda) = \lambda \arctan\left(\frac{1}{\lambda}\right) + \frac{1}{2}\log\left(1 + \lambda^2\right). \tag{70}$$

Higher rank spin groups

Now imagine the spin operators $\vec{\mathbf{S}}_a$ are realized as the generators of some simple Lie algebra $\mathbf{k} = \mathrm{Lie}K$. Let $r = \mathrm{rank}(\mathbf{k})$. The number of spin sites L and the excitation level N of our previous models generalize to the vectors: $\vec{L} = (L_1, L_2, \dots, L_{\mathbf{r}})\,, \vec{N} = (N_1, N_2, \dots, N_{\mathbf{r}})$. The twist parameter becomes the r-tuple of angles: $(\vartheta_1, \dots, \vartheta_{\mathbf{r}})$, which define the element of the maximal torus of K. The Bethe equations read as follows:

$$\begin{aligned} &\prod_{a=1}^{L_{\mathbf{i}}} \frac{\lambda_j^{(\mathbf{i})} - \nu_a^{(\mathbf{i})} + is_a^{(\mathbf{i})} u}{\lambda_j^{(\mathbf{i})} - \nu_a^{(\mathbf{i})} - is_a^{(\mathbf{i})} u} = \\ &= e^{i\vartheta_{\mathbf{i}}} \prod_{\mathbf{j}=1}^{\mathbf{r}} \prod_{j:\,(i,\mathbf{i})\neq(j,\mathbf{j})} \frac{\lambda_i^{(\mathbf{i})} - \lambda_j^{(\mathbf{j})} + i\mathscr{C}_{\mathbf{ij}} u/2}{\lambda_i^{(\mathbf{i})} - \lambda_j^{(\mathbf{j})} - i\mathscr{C}_{\mathbf{ij}} u/2} \end{aligned} \tag{71}$$

where the unknowns (Bethe roots) are $\lambda_i^{(\mathbf{i})}$, $\mathbf{i} = 1, \dots, r$, $i = 1, \dots, N_{\mathbf{i}}$. The equations (71) describe the spectrum of the transfer matrix acting in the space

$$\mathscr{H}_{\vec{L}} = \bigotimes_{\mathbf{i}=1}^{\mathbf{r}} \otimes_{a=1}^{L_{\mathbf{i}}} \mathscr{W}_{s_a^{(\mathbf{i})}}^{(\mathbf{i})} \left(\nu_a^{(\mathbf{i})}\right)$$

where $\mathscr{W}_s^{(i)}(\nu)$, $2s \in \mathbf{Z}_{\geq 0}, \nu \in \mathbf{C}$ are the so-called Kirillov-Reshetikhin modules, the special evaluation representations of the Yangian $\mathscr{Y}(\mathbf{k})$ of $\mathbf{k}$. The matrix $\mathscr{C}_{\mathbf{ij}}$ in (71) is the Cartan matrix of $\mathbf{k}$.

The equations (71) also have a YY function, see [19] for details. The most general closed spin chains correspond to yet more general representations of the Yangian $Y(\mathbf{k})$, not necessarily the Kirillov-Reshetikhin ones. These representations $\mathscr{W}_{\vec{\mathbf{P}}}$ are characterized by the highest weights, which are given by an r-tuple $\vec{\mathbf{P}}$ of monic polynomials, called Drinfeld polynomials:

$$\vec{\mathbf{P}} = (P_1(\lambda), P_2(\lambda), \dots, P_{\mathbf{r}}(\lambda)) \tag{72}$$

For example, in the case of $\mathbf{k} = \mathbf{sl}_2$, the inhomogeneous spin chains were characterized by the polynomials $a(\lambda)$ and $d(\lambda)$. These polynomials enter Baxter's equations (65). These two polynomials can be related to the single Drinfeld polynomial $P_1(\lambda)$, as it should be, since the rank of $\mathbf{sl}_2$ is equal to one:

$$\frac{a(\lambda)}{d(\lambda)} = \frac{P_1(\lambda + \frac{1}{2})}{P_1(\lambda - \frac{1}{2})} \tag{73}$$

Explicitly ($\hat{s}_a = s_a - \frac{1}{2}$):

$$P_1(\lambda) = \prod_{a=1}^{L} \prod_{m_a=-\hat{s}_a}^{\hat{s}_a} (\lambda - \nu_a + i m_a) \tag{74}$$

In the general case the Bethe roots again form $\mathbf{r}$ groups $\left(\lambda_i^{(\mathbf{i})}\right)$, $\mathbf{i} = 1, \ldots, \mathbf{r}$, $i = 1, \ldots, N_{\mathbf{i}}$. The general Bethe equations can be written for the simply-laced $\mathbf{k}$, for each $(\mathbf{i}, i)$, as:

$$\frac{P_{\mathbf{i}}(\lambda_i^{(\mathbf{i})} + \frac{i}{2})}{P_{\mathbf{i}}(\lambda_i^{(\mathbf{i})} - \frac{i}{2})} = e^{i\vartheta_{\mathbf{i}}} \prod_{\mathbf{j}=1}^{\mathbf{r}} \prod_{j=1}^{N_{\mathbf{j}}} \frac{\lambda_i^{(\mathbf{i})} - \lambda_j^{(\mathbf{j})} + \frac{i}{2}\mathscr{C}_{\mathbf{ij}}}{\lambda_i^{(\mathbf{i})} - \lambda_j^{(\mathbf{j})} - \frac{i}{2}\mathscr{C}_{\mathbf{ij}}} \tag{75}$$

There exists also the generalizations to the non-simply laced $\mathbf{k}$, and some partial results for the affine case as well, see [19] for details and references.

The equations (75) can be also written in the form of Baxter-like equations for $\mathbf{r}$ polynomial functions $\mathbf{Q}_{\mathbf{i}}(\lambda) = \prod_{i=1}^{N_{\mathbf{i}}} (\lambda - \lambda_i^{(\mathbf{i})})$, either directly using (75), see [19], or using the theory of q-characters [45], or, for $\mathbf{k} = su(\mathbf{r}+1)$, using the discrete Hirota equations [46].

Anisotropic chains

The model with the (58) Hamiltonian can be generalized to the anisotropic situations:

$$H_{\text{Heis}} = \sum_{a=1}^{L} \left(J_x S_a^x S_{a+1}^x + J_y S_a^y S_{a+1}^y + J_z S_a^z S_{a+1}^z \right) \tag{76}$$

with the general anisotropy parameters J_x, J_y, J_z. These more general spin chains (the XXZ, XYZ, or the 8-vertex model [47]) also admit the Bethe ansatz, with the Bethe equations (63) replaced by the trigonometric or elliptic analogues.

THE DICTIONARY

In this section we present the explicit bridge between the two topics of our story, the dictionary, relating the quantum integrable spin chains and the $\mathscr{N} = (2,2)$ supersymmetric gauge theories in two dimensions.

We do it here at the example of the inhomogeneous twisted XXX_s spin chain and a certain $U(N)$ gauge theory in two dimensions. This map extends to other examples presented above and more, see [19] for details.

The foundation of our dictionary is of course the observation that the vacuum equation for the gauge theory (1) coincides with Bethe equation in the integrable theory (which we formulate in some generality in (63), (75)):

The effective twisted superpotential corresponds to the YY function

Actually, the entries of the YY function are dimensionless, while the vacuum equation (1) is written for σ, which has the dimension of mass. The precise relation reads as follows:

$$\boxed{\begin{aligned} uY(\lambda u) &= \tilde{W}^{\text{eff}}(\sigma) \\ \lambda_i u &= \sigma_i \end{aligned}} \tag{77}$$

where u is the particular twisted mass, corresponding to the $U(1)$ symmetry breaking the $\mathscr{N} = 4$ supersymmetry of the theory we present below, down to $\mathscr{N} = 2$.

Of course this is only a starting point leading to precise identification of two theories – the vacuum structure, including the vacuum expectation values of the (twisted) chiral operators on the gauge theory side and the entire spectrum of all integrable Hamiltonians on the spin chain side. The Baxter operator(s) $\mathbf{Q}_{\mathbf{i}}(\lambda)$ are identified, up to the rescaling $\lambda \to x = \lambda u$, $Q_{\mathbf{i}}(\lambda) \to u^{-N_{\mathbf{i}}} Q_{\mathbf{i}}(x)$, with the BC order parameters of the gauge theory.

The $\tilde{Q}\Phi Q$ theory vs the XXX_s spin chain

Our announced duality maps the inhomogeneous XXX_s spin chain to the $U(N)$ gauge theory with the following matter fields and twisted masses:

Gauge representation	Matter multiplets	Twisted mass
adjoint	Φ	$m^{\text{adj}} = -iu$
$\mathbf{N}$	Q_a	$m_a^{\text{f}} = -\mu_a + i s_a u$
$\bar{\mathbf{N}}$	$\tilde{Q}^a$	$m_a^{\bar{\text{f}}} = \mu_a + i s_a u$
	$a = 1, \ldots, L$	

(78)

In the absence of superpotential all the parameters are complex numbers, $\mu_a, s_a, u \in \mathbf{C}$. The generic superpotential (17) breaks the global symmetry group $U(L) \times U(L) \times U(1)$ down to the subgroup $U(1)$ of the transformations $Q_a \mapsto e^{i\mu_a} Q_a$, $\tilde{Q}^a \mapsto e^{-i\mu_a} \tilde{Q}^a$. However, if the matrix-valued function $m_a^b(\Phi)$ is chosen in a special way, the unbroken subgroup gets enhanced. In particular, when

$$m_a^b(\Phi) = \delta_a^b \varpi_a \Phi^{2s_a}\,, \tag{79}$$

for $a, b = 1, \ldots, L$, for some complex constants ϖ_a, we have the group $U(1)^L \times U(1)$ of the transformations of the form:

$$Q_a \mapsto e^{i\mu_a - i s_a u} Q_a\,,\ \tilde{Q}^a \mapsto e^{-i\mu_a - i s_a u} \tilde{Q}^a\,, \Phi \mapsto e^{iu} \Phi \tag{80}$$

In this case we turn on, in addition to the superpotential:

$$W_{\tilde{Q}\Phi Q} = \sum_{a=1}^{L} \varpi_a \tilde{Q}^a \Phi^{2s_a} Q_a \tag{81}$$

the twisted masses (78) with the non-negative integer $2s_a$, in order for the superpotential (17) to be a polynomial. Note that the $\mathcal{N} = 2$, $d = 4$ theory has a superpotential given by: $W_0 = \sum_{a=1}^{L} \tilde{Q}^a \Phi Q_a$ (in the massless case) and corresponds to $s_a = \frac{1}{2}$.

A few comments about the superpotential (17) are in order. In two dimensions the corresponding theory is renormalizable for all half-integer values of s. In three dimensions only for $s = \frac{1}{2}$ or $s = 1$ we get renormalizable theory, and in four dimensions - only for $s = \frac{1}{2}$. One has several approaches to the three and four dimensional theories for the values of s when the superpotentials $\tilde{Q}\Phi^{2s}Q$ are not renormalizable: 1.$\rangle$ Think about these theories as effective theories arising from a renormalizable fundamental theory after integrating out some massive modes; 2.$\rangle$ View them as the theories with cutoff; 3.$\rangle$ Embed them into string theory, or 4.$\rangle$ Abandon them for such values of s altogether. Obviously we do not like to pursue the last option. We describe the details of 1.$\rangle$ in [19].

> Thus, the $\tilde{Q}\Phi Q$ theory with the superpotential (81) and the twisted masses (78) with the half-integers s_a is mapped to the N-particle sector of the twisted inhomogeneous $SU(2)$ XXX_s spin chain. The supersymmetric vacua correspond to Bethe states. The twisted masses correspond to the inhomogeneities and the local spins. The Fayet-Illiopoulos term combined with the theta angle map to the complexified twist parameter of the spin chain.

Indeed, since the gauge group $U(N)$ has a center, one has an additional parameter, the complexified theta angle, which is the sum of the theta angle and the Fayet-Illiopoulos term:

$$t = \frac{1}{2\pi}\vartheta + ir \tag{82}$$

This parameter is mapped to the twist parameter of the (complexified) spin chain:

$$\boxed{t = \frac{1}{2\pi}\vartheta + ir \longrightarrow \vec{S}_{a+L} = e^{-\pi i t \sigma_3} \vec{S}_a e^{\pi i t \sigma_3}} \tag{83}$$

Note that the sole rôle of the superpotential W (81) is to impose the integrality condition on the s_a parameters of the twisted masses (78). It is conceivable that in the absence of W the theory with complex s_a's maps to the $\mathbf{sl}_2$ spin chain with possibly infinite dimensional spin representations (still in the N-particle sector).

Order parameters, Hamiltonians, local operators

Let us discuss the rôle of the BC order parameter and Baxter's equation in the gauge theory. Define the gauge theory observable, which we shall call the T-operator (cf. (46)):

$$\mathbf{T}(x) = a(x)\frac{\mathbf{Q}(x+m^{\mathrm{adj}})}{\mathbf{Q}(x)} + e^{2\pi i t}\, d(x)\frac{\mathbf{Q}(x-m^{\mathrm{adj}})}{\mathbf{Q}(x)} \tag{84}$$

with $a(x), d(x)$ from (47). The T-operator is an infinite expansion in x, whose coefficients are the gauge invariant functions of σ. In a sense, we can view $t(x)$ as the generating function of the twisted chiral ring operators. Now, the twisted chiral ring is a commutative associative ring [19] generated by the coefficients of $\mathbf{Q}(x)$, and the relations which can be concisely formulated as:

$$\mathbf{T}(x)_{-} \equiv \sum_{n=1}^{\infty} T_n x^{-n} = \{\mathcal{Q}, \ldots\} \tag{85}$$

where $\mathcal{Q}$ is one of the supercharges of the theory. In other words, in the twisted chiral ring the following equations hold:

$$T_n = 0\,, \quad n = 1, 2, \ldots \tag{86}$$

It would be nice to derive this from some Ward identities, analogous to the generalized Konishi anomaly [48]. In the spin chain the positive coefficients of the expansion of $t(x)$ correspond to the integrable Hamiltonians H_k of the model:

$$\mathbf{T}(x)_{+} = (1+e^{2\pi i t})x^L + \sum_{a=1}^{L} H_{n-1}x^{n-1} \tag{87}$$

Finally, the gauge theory has non-local operators, creating soliton states, interpolating between different vacua of the theory. It is natural to identify those with local operators in the spin chain, such as the operator of the local spin $\vec{\mathbf{S}}_a$.

LIFTS TO HIGHER DIMENSIONS

Our two dimensional theories can be lifted to three and four dimensions while keeping the same amount of supersymmetry. The three dimensional theory compactified on a circle would map to the XXZ spin chain (cf. (48) with (78)), the four dimensional theory compactified on $\mathbf{T}^2$ (cf. (54)) maps to the 8-vertex model and the XYZ spin chain.

From four dimensional $\mathcal{N}=2$ susy to the two dimensional $\mathcal{N}=2$ susy

There is, however, another four-dimensional construction which leads to an interesting deformation of the would-be-Bethe equations, which are the vacuum equations of the compactified four-dimensional supersymmetric gauge theory. We start with the $\mathcal{N}=2$ supersymmetric gauge theory in four dimensions and compactify it on a two-dimensional sphere $\mathbf{S}^2$. Of course, this compactification breaks supersymmetry, so we shall have to make a partial twist along $\mathbf{S}^2$ to preserve some fraction of the supersymmetry.

This theory is interesting as the low-energy two dimensional dynamics is sensitive to the effects of the four dimensional instantons. The equations (34) then contain the complexified four-dimensional coupling

$$\mathcal{T}=\frac{\theta}{2\pi}+\frac{4\pi i}{e^2} \tag{88}$$

and, for the appropriate four dimensional theory, are modular.

The partial twist is done as follows (cf. [49]). The holonomy group of the product manifold $\Sigma\times\mathbf{S}^2$ with the product metric is $SO(2)_\Sigma\times SO(2)_{\mathbf{S}^2}$. Here Σ is the worldsheet of the effective two dimensional theory. In addition, the $\mathcal{N}=2$ theory has an $SU(2)$ R-symmetry group (it can be larger for the theories with matter). The supercharges of the $\mathcal{N}=2$ theory, eight of them, transform as $\left(\pm\frac{1}{2},\pm\frac{1}{2},\mathbf{2}\right)$ under $SO(2)_\Sigma\times SO(2)_{\mathbf{S}^2}\times SU(2)$. Since the two-sphere has no covariantly constant spinors, none of these supercharges are conserved, if the R-symmetry group is to be preserved. Now imagine $SO(2)_{\mathbf{S}^2}$ is allowed to act on the R-symmetry index. In other words, let us embed $SO(2)_{\mathbf{S}^2}\to SU(2)$, via

$$e^{i\alpha}\mapsto\begin{pmatrix} e^{iq\alpha} & 0\\ 0 & e^{-iq\alpha}\end{pmatrix},\ 2q\in\mathbf{Z} \tag{89}$$

The eight supercharges now transform as: $\left(\pm\frac{1}{2},\pm\frac{1}{2}\pm q\right)$ under $SO(2)_\Sigma\times SO(2)_{\mathbf{S}^2}$. We now can choose $q=\pm\frac{1}{2}$, to make four supercharges have vanishing charge under $SO(2)_{\mathbf{S}^2}$. The other four supercharges transform as: $\left(\pm\frac{1}{2},\pm1\right)$ and are not conserved on the two-sphere $\mathbf{S}^2$.

Pure $\mathcal{N}=2$ theory

As a warmup, consider the compactification of the pure $\mathcal{N}=2$ super-Yang-Mills theory on $\mathbf{S}^2$ with the $q=\pm\frac{1}{2}$ twist.

The result is the two dimensional theory, with the $\mathcal{N}=2$ supersymmetry in two dimensions. The field content of that theory contains a massless vector multiplet and a Kaluza-Klein tower of massive vector and chiral multiplets, all transforming in the adjoint representation of the gauge group. The lowest massive level comes from the Laplacian eigenstates in the space of the one-forms on $\mathbf{S}^2$.

Now we wish to calculate the effective twisted superpotential of the two dimensional theory. We shall take the size of $\mathbf{S}^2$ to zero. In this way the massive states become infinitely massive and ought to decouple.

Now let us turn on the magnetic flux on the two-sphere. More precisely, we can turn on the flux, for $G=U(N)$,

$$\frac{1}{2\pi i}\int_{\mathbf{S}^2}F\sim\mathrm{diag}\left(\mathbf{m}_1,\ldots,\mathbf{m}_N\right)\ ,\ \mathbf{m}_i\in\mathbf{Z}$$

in the maximal torus of the gauge group, determined by the vacuum expectation value of the adjoint Higgs field. In the presence of the magnetic flux, some of the charged Kaluza-Klein modes become massless and contribute to the effective twisted superpotential. As a result, the twisted superpotential can be expressed in terms of the prepotential of the four dimensional theory as follows:

$$W(a)=\sum_{i=1}^{r}\mathbf{m}_i\frac{\partial\mathcal{F}}{\partial a^i} \tag{90}$$

where $r=N$ for $G=U(N)$, $r=N-1$ for $G=SU(N)$ (in the latter case there is one more subtlety related to the possibility to turn on the discrete magnetic flux $w_2\in\mathbf{Z}_N$). In addition, the unfolding of the two dimensional field strength can be accomplished, as in (33), by introducing the integral vector $(\mathbf{n}_1,\ldots,\mathbf{n}_r)$, which can be identified with the vector or electric fluxes through the two-sphere. The twisted superpotential becomes [37]:

$$W(a)=\sum_{i=1}^{r}\left(\mathbf{m}_i\frac{\partial\mathcal{F}}{\partial a^i}+\mathbf{n}_i a^i\right)=\oint_{C_{\mathbf{m},\mathbf{n}}}\lambda \tag{91}$$

where $\lambda=p\,dz$ is the Seiberg-Witten differential, and $C_{\mathbf{m},\mathbf{n}}\in H_1(\mathcal{C},\mathbf{Z})$ is a cycle on the Seiberg-Witten curve $\mathcal{C}$,

$$\Lambda^N\left(e^p+e^{-p}\right)=z^N+u_1z^{N-1}+\ldots+u_N\ , \tag{92}$$

corresponding to the charges $(\mathbf{m},\mathbf{n})$.

The $\mathcal{N}=2^*$ theory

Now, to make things interesting let us add some matter fields. One of the most beautiful gauge theories in four dimensions is the so-called $\mathcal{N}=2^*$ theory. This is the $\mathcal{N}=2$ theory with massive adjoint hypermultiplet. In the ultraviolet this is the $\mathcal{N}=4$ theory, which exhibits S-duality. In the infrared this is the Abelian theory with the moduli space of vacua described by the algebraic integrable system [13], an elliptic Calogero-Moser system, which can also be described [5, 14] as a degenerate case of the Hitchin system [2]. The classical elliptic Calogero-Moser system describes the system of particles $q_1, q_2, \ldots, q_N$ on a circle, interacting via a pair-wise potential

$$U = m^2 \sum_{i,j=1}^{N} \wp(q_i - q_j)$$

which is doubly periodic, with the periods 1 and $\mathscr{T}$, $\mathrm{Im}\mathscr{T} > 0$, where we use the elliptic modulus defined by the gauge couplings (88). The classical motion of that system is mapped to the constant velocity motion on the Jacobian variety of the spectral curve,

$$\det_{N\times N}\left(\Phi(z) - \lambda\right) = 0 \tag{93}$$

where

$$\Phi_{ij}(z) = p_i \delta_{ij} + m \frac{\vartheta_1(z + q_i - q_j)\vartheta_1'(0)}{\vartheta_1(z)\vartheta_1(q_i - q_j)}(1 - \delta_{ij}) \tag{94}$$

This family of curves encodes [50, 51] the low-energy effective action of the $\mathscr{N} = 2^*$ theory with the mass of the hypermultiplet equal to m. The prepotential $\mathscr{F}$ depends on the vacuum expectation values $\langle \phi \rangle = \mathrm{diag}(a_1, \ldots, a_N)$ of the scalars in the vector multiplet of the $U(N)$ gauge group, and on m and τ:

$$\mathscr{F}(a; m, \mathscr{T}) = \mathscr{F}^{\mathrm{pert}}(a; m, \mathscr{T}) + \sum_{k=1}^{\infty} e^{2Nk\pi i \mathscr{T}} \mathscr{F}_k(a; m)\,, \tag{95}$$

where

$$\begin{aligned}\mathscr{F}^{\mathrm{pert}}(a; m, \mathscr{T}) &= \frac{\mathscr{T}}{2}\sum_{i=1}^{N} a_i^2 + \\ &+ \frac{1}{4}\sum_{i\neq j}\left[(a_i - a_j)^2\left(\log(a_i - a_j) - \frac{3}{2}\right) - \right. \\ &\left. - (a_i - a_j + m)^2\left(\log(a_i - a_j + m) - \frac{3}{2}\right)\right]\end{aligned} \tag{96}$$

The terms $\mathscr{F}_k(a;m)$ come from the charge k instantons and can be computed for any k using localization techniques [50]:

$$\mathscr{F}_1(a; m) = m^2 \sum_{i=1}^{N} \prod_{j\neq i}\left(1 - \frac{m^2}{(a_i - a_j)^2}\right),\ \text{etc.} \tag{97}$$

Now let us apply the same procedure to the $\mathscr{N} = 2^*$ theory, i.e. let us compactify the theory on a two-sphere with the partial twist. Actually, the theory with adjoint hypermultiplet can be twisted in many ways. Indeed, we have an extra $U(1)$ symmetry under which the complex scalars B_1, B_2 in the adjoint hypermultiplet have charges $+1, -1$. By embedding $SO(2)_{\mathbf{S}^2}$ into this $U(1)$ we shall assign the additional Lorentz spins to the bosons and fermions in the hypermultiplet.

The two dimensional twisted superpotential now contains, in addition to the terms (91), the terms coming from the extra twist of the matter fields (we identify $a^i = \sigma_i$):

$$\begin{aligned}\tilde{W}^{\mathrm{eff}}(\sigma; m, \mathscr{T}) &= 2\frac{\partial \mathscr{F}(\sigma; m, \mathscr{T})}{\partial m} + \\ &+ \sum_{i=1}^{r}\left(\mathbf{m}_i \frac{\partial \mathscr{F}(\sigma; m, \mathscr{T})}{\partial \sigma_i} + \mathbf{n}_i \sigma^i\right)\end{aligned} \tag{98}$$

and vacuum equation is defined with this and (1). We note that the perturbative limit of the (98) gives the twisted effective superpotential of the Yang-Mills-Higgs theory of [1, 3, 4] (the example (44) of the Hitchin theory above). This is not surprising since in the trivial instanton sector the reduction on $\mathbf{S}^2$ of the four dimensional $\mathscr{N} = 2^*$ theory gives the two dimensional $\mathscr{N} = 2^*$ theory. We see here that the four dimensional instation corrections give a modular deformation of the effective twisted superpotential, and accordingly the Bethe equations. This is a very interesting phenomenon which needs further investigation, see [19] for details.