Amplitudes in N=4 SYM from the Quantum geometry of the Momentum Space

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INTRODUCTION

The $N = 4$ SYM theory provides a possibility to recognize some features of the theories with less amount of SUSY. While $N = 4$ SYM is far from the QCD-like theories in the infrared because of the lack of confinement it shares common features in UV region where physics in asymptotically free theories is described within a perturbation theory. That is considering the perturbative expansion in $N = 4$ SYM coupling constant which does not run we could try to clarify some generic properties of the perturbative expansion in the gauge theories.

It is of the prime importance to discover any hidden symmetries at the high energies or equivalently hidden integrable structures providing the nontrivial conservation laws restricting the form of the scattering amplitudes. In the four-dimensional setup the integrability behind the amplitudes is known only at the Regge limit when the SL(2,C) spin chain gets materialized [1, 2](see [3] for review).

The simplest objects at generic kinematics are the MHV amplitudes which are the perfect starting point for any discussion since at the planar limit they can be described in terms of the single kinematical function. Even at the tree level MHV amplitudes [4] enjoy some remarkable properties. They are localized on the complex curves in the twistor space [5] and can be described as the correlators of chiral bosons on the genus zero Riemann surface [6]. It turns out that the generating function for the tree MHV amplitudes is just the particular solution to the self-duality equation in YM theory [9, 10]. It substitutes the naive superposition of the plane waves of the same chirality in a nonlinear theory. Moreover this solution provides the symplectic transformation [11](see also [12]) of the YM theory in the light-cone gauge formulation into the so-called tree MHV Lagrangian formulated in [8] which to some extend is the analogue of the t'Hooft effective vertex generated by instantons. However this approach becomes less clear when going to higher loops. Indeed, the attempt to formulate the one-loop MHV amplitudes in a twistor-like manner was not successful [7] and certainly calls for additional insights on the problem.

More recently Bern, Dixon and Smirnov (BDS) have formulated the conjecture [16] that all-loop MHV amplitudes get exponentiated and factorize into IR divergent and finite parts. Moreover it was conjectured that the finite part of all-loop amplitude involves only all-loop cusp anomalous dimension $\Gamma_{cusp}(\alpha)$ and finite part of one-loop amplitude. Inspired by this conjecture Alday and Maldacena have calculated the amplitude at strong coupling regime via minimal surfaces in AdS-type geometry with the proper boundary conditions [14]. They have found unexpected relation between the MHV amplitudes in planar limit of $N = 4$ SYM theory and Wilson polygons in the momentum space.

The Wilson polygon-amplitude duality refreshes the problem but deserves for the explanation itself. It was originally formulated at strong coupling when the Wilson loop is calculated in terms of minimal surface in the AdS_5 geometry upon a kind of T-duality transform. Later it was shown that duality holds true at the perturbative regime as well [19] which puts it on more firm ground. The important point was the formulation of the anomalous Ward identities for the special conformal transformations with respect to the dual conformal group. It fixes the kinematical dependence of the amplitudes up to five external legs [19]. However Ward identities tell nothing about the functional form of the amplitudes starting from six external legs. Recently the dual superconformal group was identified as the symmetry of the worldsheet theory of the superstring in $AdS_5 \times S^5$ geometry [38, 39].

Finally it was recognized that BDS anzatz fails at weak coupling at two loop level for six external legs [15, 18] and at strong coupling for infinitely large number of external legs. Moreover the BDS anzatz seems not fit well with the Regge limit [41]. On the other hand at two loop level the duality between Wilson polygon and MHV amplitude survives.

There are a lot of pressing questions to be answered. Just mention a few;

- Is there some geometrical picture behind the BDS anzatz which would suggest the way of its necessary generalization?

- Is there the generalization of the dual conformal Ward identity which would fix the functional form of the one-loop amplitude for any number of external legs?
- Is there the fermionic representation for the loop amplitudes which would imply the hidden integrability?
- What is the origin of the Wilson polygon - amplitude duality ?
- Is there clear geometrical picture behind the reggeization of the gluon?

To some extend we shall try to generalize the geometrical picture for the tree amplitudes suggested in [5]. At the tree level in [5] the Euclidean D1 "instanton" branes with the attached open strings have been considered in the twistor space. The D1 brane is localized at the point in the Minkowski space in agreement with the locality of the vertex generating tree MHV amplitude in the MHV formalism. To describe the loop amplitudes we shall adopt a little bit different picture and consider C^4 manifold in the B model as a "twistor-like" manifold for the complexified Minkowski space. The D3 branes substitute "D1 instantons" and are embedded in C^4. The somewhat similar objects were also introduced as the IR regulator branes in the Alday-Maldacena calculation. Indeed, it was shown [14] that dilaton field gets changed upon the T-duality in the RG radial coordinate which means that D-instanton is added to the background. After the Fourier transform along flat four-dimensions D-instanton gets transformed into the D3 brane we shall work with. The Wilson polygon which corresponds to the boundary of the string worldsheet and is presumably dual to the amplitude is located just on these IR regulator branes. Contrary to the previous considerations the positions of the regulator branes are not free but determined dynamically in terms of the cross-ratios of the external momenta.

The physics of the scattering at the loop level can be treated from the different perspectives. From the point of view of the KS gravity on the moduli space we are calculating the correlator of the fermions or the fermionic currents which can be identified with the tau-function of the 2d integrable system. The second viewpoint concerns the 4d gauge theory on the regulator branes whose number is fixed by the number of external particles. Finally one could consider the worldsheet viewpoint where the regulator branes provide the proper boundary conditions for the string. These viewpoints are complimentary and allow to check the self-consistency of our approach.

Within the KS perspective we shall discuss the fermionic representation behind the loop MHV amplitudes which would generalize the Nair's fermionic representation for the tree amplitudes. The fermionic picture is a heart of the integrability which admits the representation in terms of the chiral fermions on the Riemann surface in the external gauge field. The gauge field on the Riemann surface represents the "point of Grassmanian" or in physical terms the particular Bogolyubov transformation between the fermionic vacua. This approach was summarized in [23]. It was argued that fermions in the KS gravity correspond to mirror of Lagrangian branes in the A model. These branes are also refereed to as Kontsevich or noncompact branes and their positions on the Riemann surface yield the "times" in the corresponding integrable systems. Note that in the framework of the topological strings in A-model we discuss the Kahler geometry while in B-model the complex geometry is captured by the KS [13] theory.

The fermion one-point function corresponds to the Baker-Akhiezer function in the integrable system framework and to the single regulator brane insertion at some point on the moduli space. Since generically we are interested in the quantum integrable system the Riemann surface gets quantized and yields the corresponding Baxter equation [26]. The semiclassical solutions to the Baxter equation which are the generating functions for the Lagrangian sub-manifolds in the particular integrable system play important role in the analysis. They serve as the building blocks for the correlators in the $N = 4$ YM theory and can be considered as the "semiclassical D3 brane wave function" or as the effective action in the 4d gauge theory on the brane worldvolume. From the moduli space viewpoint the solution to the Baxter equation provides the generating function of the Lagrangian sub-manifold. The natural integrable system on the moduli space can be identified with the 3-KP system however similar to the N=2 SYM one could expect the pair of integrable system - 2D field theory and finite dimensional one. The natural finite dimensional integrable system which is responsible for the hidden symmetries at the generic kinematics is conjectured to be related to the Faddeev-Volkov model [32] and the corresponding statistical model [31] based on the discrete quantum conformal transformations.

Since we are trying to sum the perturbation series the YM coupling constant is expected to be involved into some algebraic structure behind the all-loop answer. It is this hidden symmetry which provides the choice of the particular solution to the Yang-Baxter equation. The Faddeev-Volkov solution to the Yang-Baxter implies that we are actually trying to relate the YM coupling constant with the parameter q of $U_q(SL(2,R))$. The proper identification turns out to be nontrivial problem since in particular it has to respect the S-duality group in N=4 theory. It will be argued that the BDS anzatz corresponds to the limit $q \to 1$ while the Regge limit seems to be related to the opposite "strong coupling regime" of the quantum group.

The consideration of the four-dimensional theories on the regulator brane worldvolume is useful as well. The theory is in the Coulomb phase and the position of the regulator brane on the particular Riemann surface corresponds to the coordinate on the Coulomb moduli space. Since all regulator D3 branes are at different positions on the moduli space the theory generically has the gauge group $U(1)^k$ where k is related to the number of the external gluons . The effective action of each U(1) gauge theory plays the role of the wave function of the two-dimensional fermions in KS gravity. A little bit surprisingly one has to consider not the real part of the effective action in the external field but the imaginary one involving dilogarithm. This is natural from the Euclidean viewpoint while in the Minkowski space we actually consider the probability of the pair production.

It is important to discuss separately the special Regge kinematical region were the hidden symmetries of the amplitudes where found for the first time. The hidden symmetries were captured at one loop by the SL(2,C) spin chains [1, 2]. It was shown in [35] that the N-reggeon dynamics belongs to the same universality class as conformal N=2 SQCD with $N_f = 2N$ at the strong coupling orbifold point. We shall argue that the brane geometry in the reggeon case is similar to the one in SQCD which provides the qualitative explanation of the same universality class for both theories. The new object is the open string stretched between two regulator branes and is the analogue of the massive vector bosons in the conventional N=2 SYM theory. Here we shall tempt to interpret these open strings as the "reggeons". The masses of these effective degrees of freedom correspond to the differences of the positions of the regulator branes on the Riemann surface.

THE LOOP RESULTS FOR THE MHV AMPLITUDES

Let us remind the main results concerning the loop MHV amplitudes. The MHV gluon amplitudes involve two gluons of the negative chiralities and the rest of gluons have positive chiralities. Consider the ratio of all-loop and tree answers. The following form of the all-loop amplitudes has been suggested in [16]

$$log(\frac{M_{all=loop}}{M_{tree}}) = (F_{div} + \Gamma_{cusp}(\lambda) M_{one-loop}) \qquad (1)$$

which involves only the all-loop answer for the cusp anomaly Γ_{cusp} and one-loop MHV amplitude. The IR divergent part F_{div} gets factorized in the all-loop answer. The cusp anomaly measures UV behavior of the contour with cusp [24]. Recently the closed integral equation has been found for the cusp anomalous dimension in $N = 4$ SYM theory [40] which correctly reproduces the weak and strong coupling expansions.

The finite part of the one-loop MHV which presumably defines the all-loop answer can be written in terms of the finite part of the so-called two-mass easy box function F^{2em} [34]

$$M_{one-loop,finite} = \sum_{p,q} F^{2em,f}(p,q,P,Q) \qquad (2)$$

This function can be expressed in terms of the dilogarithms only

$$F^{2em,f}(p,q,P,Q) = Li_2(1-aP^2) + Li_2(1-aQ^2) - Li_2(1-a(q+P)^2) - \qquad (3)$$

where

$$a = \frac{P^2+Q^2-(q+P)^2-(p+P)^2}{P^2Q^2-(q+P)^2(q+P)^2} \qquad (4)$$

and $p+q+P+Q = 0$. One more expression for the function $F^{2em,f}$ which will be useful later can be written in terms of the variables $x_{i,k} = p_i - p_k$ in terms of the sums [19]

$$\sum_i \sum_r Li_2(1 - \frac{x^2_{i,i+r} x^2_{i=1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}) \qquad (5)$$

where

$$x_i = p_{i+1} - p_i \qquad (6)$$

Since all external momenta are on the mass shell the arguments of dilogarithms are expressed in terms of the cross-ratios of the scalar products of the momenta only.

The BDS anzatz (1) has been checked at weak and strong coupling regimes. At strong coupling analyzed in the stringy setup [14] one considers first the T-duality transformation on the worldsheet which effectively interchanges UV and IR regions in the AdS_5 geometry. Then the calculation of the amplitude reduces to the calculation of the minimal surface in the dual AdS space bounded by the polygon formed by the external on-shell gluon momenta. For the four external legs the answer fits with the BDS anzatz for all-loop amplitude.

It was conjectured in [14] that any MHV N-leg amplitude follows from the vacuum expectation value of the Wilson loop of the special form

$$\frac{M_{all-loop}}{M_{tree}} = < W(p_1, p_2, ..., p_N) > \qquad (7)$$

where the closed Wilson loop polygon has light-like momenta at the edges and vertexes at x_i. Its closeness is provided by the total momentum conservation.

At weak coupling to check this polygon-amplitude duality one considers the expansion of the Wilson polygon in the YM coupling treating Wilson loop as one in the coordinate space. The perfect matching of Wilson loop and amplitudes has been found for one- and two loop answers up to six external legs [15, 18]. Moreover it was demonstrated that the anomalous Ward identities for the special conformal transformations of the form

$$K^{\nu} W(x_1, \dots x_N) = \sum_{i=1}^{n} (2x_i^{\nu} x_i \partial_i - x_i^2 \partial_i^{\nu}) W(x_1, \dots x_N) = \frac{1}{2} \Gamma_{cusp} \sum_{i=1}^{n} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1} \quad (8)$$

fix the answer up to four external legs [21]. The BDS anzatz has to be modified for generic amplitude while the Wilson polygon-MHV amplitude duality has the chance to be all-loop exact.

FINITE PART OF N=4 SYM MHV AMPLITUDES AND MOMENTUM SPACE GEOMETRY

Fermionic picture

Let us now consider the four-dimensional case and formulate our proposal for finite part of the MHV loop amplitudes. Remind that the tree amplitudes were described in terms of the D1 string instanton embedded into the twistor manifold [5]. The instanton is localized at point in the Minkowski space and open strings representing gluons are attached to it. To describe the loop amplitude we shall substitute D1 brane by the IR regulator branes embedded into the proper manifold. The gluons are attached to the regulator branes whose embedding coordinates are considered as dynamical degrees of freedom. Contrary to tree case regulator branes are localized at the sub-manifold of the complexified Minkowski space.

The starting point is the representation of the N=4 theory via geometrical engineering [22] as the IIA superstring compactified on the three-dimensional Calabi-Yau manifold which was identified as the $K3 \times T^2$ geometry in the singular limit. One has to consider the singular limit of K3 manifold when it develops A_{N-1} singularity, where N becomes the rank of the gauge group, and upon blowing up procedure it can be represented as ALE_N geometry. On the other hand the Kahler class of the T^2 can be identified with the coupling constant

$$Area(T^2) = 1/g_{YM}^2 \quad (9)$$

At weak coupling the torus is large and can be approximated by the complex plane. That is the geometry can be roughly approximated by C^3 upon the particular blow-ups.

As we have seen the one-loop answer for the MHV amplitude determining the BDS form of the amplitude involves the sum of the dilogarithms depending on the cross-ratios of the x_i variables. Below we shall try to explain how such functions with cross-ratio arguments emerge naturally both in A-model and B-model frameworks. As is well-known the A-model captures the information about the Kahler moduli while the B-model about the complex moduli and we shall see where these moduli comes from. The brane description of the scattering amplitude involves the set of the Lagrangian branes in the A-model and the corresponding B-model branes. It is these branes which provide the corresponding moduli spaces.

Let us interpret the BDS anzatz in terms of the correlator of the noncompact Euclidean D3 branes embedded into the four dimensional complex space. Consider 3d complex manifold which is mirror to the topological vertex [29]. This manifold classically is described by the equation in the C^4 with coordinates x, y, u, v

$$xy = e^u + e^v + 1 \quad (10)$$

At the discriminant locus it defines the Riemann surface

$$H(v,u) = e^u + e^v + 1 = 0 \quad (11)$$

of genus zero with three different asymptotic regions. We shall argue that the loop MHV amplitudes can be identified with the fermionic correlators on the Riemann surface (11). Fermions on the surface (11) represent the degrees of freedom in the KS gravity that is the IR regulator D3 branes imbedded into C^4 geometry.

There are two D3 branes defined by the equations

$$x = 0 \quad H(v,u) = 0 \quad (12)$$

and

$$y = 0 \quad H(v,u) = 0 \quad (13)$$

which intersect along the Riemann surface. The intersecting branes provide the natural fermionic degrees of freedom on the intersection surface from the open strings stretched between these branes. The fermions are in external field amounted from the worldvolume gauge connection on the intersecting branes. In addition to two branes intersecting along the Riemann surface we introduce the set of Kontsevich -like branes classically localized at the points (v_i, u_i) at the Riemann surface. The number of such branes is fixed by the number of the external gluons and the coordinates of these branes on the surface are defined by some particular cross-ratios. At quantum level D3 branes are extended along the Lagrangian submanifold in the (u,v) space. The cross-ratios are the natural coordinates on the moduli space of the punctured spheres that is the (u,v) space is related to the $T^* M_{0,4}$.

At the next step the Riemann surface gets quantized and the branes-fermions should obey the equation of the quantum Riemann surface that is Baxter equation which provides the wave functions depending on the separated variables. The Baxter equation in our problem reads as

$$(e^{\hbar\partial_v}+e^v+1)Q(v)=0 \tag{14}$$

Its solution turns out to be the quantum dilogarithm [23]. Note that the solution to the Baxter equation in our case can not be presented in the polynomial form that is we have infinite number of the Bethe roots.

To get the MHV all-loop amplitude in the BDS form we take the semiclassical limit of the fermionic correlator on this surface. Indeed using the semiclassical limit for the quantum dilogarithm we can represent the four-point fermionic correlator as

$$<\bar{\Psi}(z_1)\bar{\Psi}(z_2)\Psi(z_3)\Psi(z_4)>\propto exp(\hbar^{-1}(Li_2(z_3)+Li_2(z_4)-Li_2(z_1)-Li_2(z_2)) \tag{15}$$

This expression exactly coincides with the expression for the contribution of the single 2-easy mass box diagram hence upon the identification of the Planck constant

$$\hbar^{-1}=\Gamma_{cusp}(\lambda) \tag{16}$$

we reproduce BDS anzatz for the finite part of the amplitude. Indeed the one-loop answer for the MHV amplitude can be expressed purely in terms of the sum of 2-mass easy box diagrams with different grouping of the gluon momenta and therefore in terms of the fermionic correlators.

Since the regulator brane (D1 "instanton") yielding the tree amplitude is localized in the complexified Minkowski space M^c [5] one could ask about similar localization of regulator branes responsible for the higher loop calculations. To this aim recall that M^c is equivalent to the Grassmanian $Gr(2,4)$. On the other hand the factor of the Grassmanian by the maximal torus action is related to the compactified moduli space [43]

$$Gr(2,4)//T=\bar{M}_{0,4} \tag{17}$$

This representation allows us to represent the complexified Minkowski space itself as the fancy divisor of the $M_{0,4}$ [42]. We suggest that this realization implies the localization of the regulator branes on the submanifold of $T^*(M^c//T)$. It is natural to identify this manifold with the Riemann surface where the KS degrees of freedom live.

Let us present the qualitative argument concerning the corresponding A-model picture. In the A-model we introduce the set of Lagrangian branes with topology $S^1\times R^2$. The emergence of the dilogarithm as the wave function of the Lagrangian brane has been discovered in the C^3 geometry in [30]. The brane/asntibrane can be considered as the insertion of the fermion/antifermion [30] in the fermionic representation of the topological vertex picture [29].

The regulator brane worldvolume theory

Since fermions in KS framework are identified as the D3 regulator branes the natural question concerns their four-dimensional worldvolume theory. The theory on the regulator branes share many features with N=2 and N=1 SYM low-energy sectors. The number of the regulator branes is fixed by the number of the external gluons so naively one could expect a kind of $SU(K)$ gauge theory. The worldsheet theory on the regulator branes enjoys the complex scalar corresponding to the complex coordinate z of the brane on the Riemann surface (11). This is similar to the situation when the vev of the scalar field corresponds to the position of the D4 branes on the a-plane in the IIA realization of the N=2 SYM theory [49].

Since the different regulator branes are at the different points on the Riemann surface we can speak about the Coulomb branch of the regulator worldvolume theory. However their positions on the Riemann surface are fixed that is we could say about the localization of the D3 branes on the points of the moduli space $M_{0,4}$. Similar to the N=1 SYM theory when branes are localized at positions corresponding to the discrete vacua the D3 regulator branes are localized at some points parameterized by the cross-ratios. These points correspond to the local rapidities in the framework of integrability and simultaneously have to correspond to the minima of the effective superpotentials $W_{eff}(z_i)$ in the regulator worldvolume theory.

Since we attributed dilogarithms to the regulator brane wave functions it is necessary to explain where they come from in the worldvolume theory. The qualitative arguments looks as follows. In the worldvolume theory there are massive excitations corresponding to the open strings stretched between two regulator branes. They are analogue of the massive W-bosons in N=1 SYM theory on the Coulomb branch. In our case the masses of these particles are related to the cross-ratios. To recover the dilog let us remind that usually in the external field the effective action develops the imaginary part corresponding to the pair creation. The probability of the pair creation on the external field is described by the classical trajectory in the Euclidean space and in the leading approximation reads as

$$ImS_{eff}\propto e^{-\frac{m^2}{eE}} \tag{18}$$

for a particle of the mass m in the external field E. Upon taking into account the multiple wrapping and the quadratic fluctuations one gets for the scalar particle Schwinger pair production

$$ImS_{eff}\propto\sum_n\frac{1}{n^2}e^{-\frac{nm^2}{eE}} \tag{19}$$

that is dilog plays the role of the decay probability. Hence one can say that we are considering the Euclidean version of the regulator worldvolume theory and the amplitude from this viewpoint is described via bounce type configuration corresponding to the creation of the pairs of the effective massive degrees of freedom. Note that the real part of the effective action corresponds to the summation over the loops of the same degrees of freedom in the loops.

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In the A-model one can similarly consider the worldvolume theory on the D2 Lagrangian regulator branes. In this case the corresponding dilog functions emerge upon summation over the disc instantons with boundaries located at the corresponding Lagrangian branes which provide the effective superpotential in the worldvolume theory

$$W_{eff} \propto \sum_n \frac{d_n}{n^2} e^{-nA} \tag{20}$$

where A -is the corresponding area of the target disc. Note that in the A model D2 branes wrapped around the ideal tetrahedrons whose Kahler classes are defined by the cross-ratios provide the masses of the same effective "W-bosons" as in B-model.

Let us comment on the identification of the Planck constant providing the quantization of the KS gravity as the inverse cusp anomalous dimension inspired by the BDS anzatz. At the first glance it looks completely groundless however the argument supporting this identification goes as follows. The emergence of the cusp anomaly in the exponent means from the worldsheet viewpoint that it plays the role of the effective string tension or equivalently the inverse Planck constant. Such effective tension emerges if one considers the string whose boundary is extended along the light-like contours. It was shown [17] that in the limit suggested in [44] the string worldsheet action can be identified with $O(6)$ sigma model and the energy of the ground state in $O(6)$ model is proportional to the length of the string multiplied by the $\Gamma_{cusp}(\alpha)$. That is indeed $\Gamma_{cusp}(\alpha)$ plays the role of the effective tension of the string in this special kinematics. Since in our case the boundary of the string worldsheet lies on the Wilson polygon the effective tension involving the cusp anomalous dimension is natural.

However certainly this point is far from being clarified. For instance in the Ward identity for the special conformal transformation Γ_{cusp} enters as the multiplier in the anomalous contribution. This claim has been explicitly checked at the first loops in the gauge theory calculations and the arguments that it holds true at all orders have been presented. This means that in the anomalous Ward identity it plays the role of the Planck constant not the inverse one. To match both arguments we could suggest that in the Ward identity we are considering the S-dual formulation and therefore the D1 string worldsheet action instead of the F1 one in $O(6)$ sigma model. This would imply that the Wilson polygon equivalent to the MHV amplitude could be considered as the boundary of the D1 string as well.

In more general setup it is highly desirable to realize the meaning of the relation of such type in the first quantized language. Since the cusp anomalous dimension is just the renormalization factor for the self-crossing of the worldline it is very interesting to understand if such self-crossing is involved into the quantization issue. In particular in the Ising model the effect of the self-crossing is captured by the topological term and in the description of the topological string on C^3 somewhat similar θ term in six dimensions plays the role of the quantization parameter indeed [28]. In the gauge theory language such objects are related to the renormalization of the double-trace operators couplings.

INTEGRABILITY BEHIND THE SCATTERING AMPLITUDES

General remarks

In this Section we shall discuss the hidden integrability behind the scattering amplitudes and present the arguments that similarly to the integrability pattern behind effective actions in N=2 SYM theory two integrable systems are involved. The degrees of freedom of both integrable systems are related to the coordinates of the regulator branes. One of these systems which we identify as the Whitham-like 3-KP one plays the role of RG flows in the regulator brane worldsheet theory or equivalently the motion of the regulator brane along the "radial" RG-coordinate. The second integrable system generalizing the Hitchin-like or spin chain models involves the effective interactions between the regulator branes. We shall give arguments that this system is based on the Faddeev-Volkov solution to the Yang-Baxter equation for the infinite-dimensional representations of the noncompact $SL(2,R)$ group.

Recall how two integrable systems are involved into the description of the low energy effective actions of N=2 SYM theories. The first finite dimensional system is of the Hitchin or spin chain type and its complex Liouville tori are identified with the Seiberg-Witten curves. This spectral curve emerges in the gauge theory upon the summation over the infinite number of instantons [36].

Following [25] one can canonically define the dual integrable system whose phase space is built on the integrals of the motion of the first one. In the simplest case of $SU(2)$ theory the phase space for the dual system has the symplectic structure [45]

$$\omega = da \wedge da_D \quad (21)$$

where the variables (a, a_D) are the standard variables in N=2 SYM framework [48]. The prepotential $\mathscr{F}$ can be identified with the generating function of the Lagrangian sub-manifold in the dual system with the a, a_D phase space

$$H(a(u), \frac{\partial \mathscr{F}}{\partial a})) = u \quad (22)$$

and obeys the Hamilton-Jacobi equation

$$\frac{\partial \mathscr{F}}{\partial \, log \Lambda} = H \quad (23)$$

In the brane setup the prepotential defines the semi-classical "wave function" of the D4 brane $\Psi(a) \propto exp(\hbar^{-1}\mathscr{F}(a))$ in the IIA brane picture where perturbatively the argument of the wave function can be identified with coordinate of the D4 brane on the NS5 brane. The total perturbative prepotential in $SU(N_c)$ can be considered as a sum of the exponential factors in the product of the wave functions of N_c D4 branes. In the A-model side these wave functions can be considered in the Kahler gravity framework and the arguments of the wave function have to be treated as the Kahler classes of the blow-upped spheres.

The integrals of motion provide the moduli space of the complex structures in the Calabi-Yau geometry in the B model hence we are precisely in the KS framework. In this B-model formulation we consider the argument of the brane wave function as the coordinate on the moduli space of the complex structures. The dual Whitham-type integrable system naturally defines the τ-function of the 2d Toda theory formulated in terms of the chiral fermions on the Riemann surface with two marked points.

3-KP system

Let us turn to the integrable structure relevant for the scattering amplitudes at generic kinematics and first identify the degrees of freedom and evolution "times". As we have described above the fermionic degrees of freedom correspond to the noncompact branes localized on the Riemann surface. The two-dimensional field theory corresponds to the reduction of the KS theory on the two-dimensional surface. The fields on the surface are in the external abelian connection of the Berry type which tells how the B- branes transform under the change of the complex structure fixed by the momenta of external particles.

The form of the Riemann surface $H(u,v) = 0$ dictates that there are three infinities and therefore we are dealing with the particular solution to 3-KP integrable system. To describe the integrable system it is convenient to introduce the chiral fermions with the following mode expansion

$$\psi(x_i) = \sum_n \psi^i_{n+1/2} x_i^{-n-1}, \qquad \psi^*(x_i) = \sum_n \psi^{*i}_{n+1/2} x_i^{-n-1} \quad (24)$$

around the i-th infinity, $i = 1,2,3$ and the commutation relations

$$\{\psi^i_n, \psi^{*j}_m\} = \delta^{ij}\delta_{n+m,o} \quad (25)$$

Defining the vacuum state by relations

$$\psi_n|0>= 0, \quad \psi^*_n|0>= 0, \quad n > 0 \quad (26)$$

the generic state $|V>$ can be presented in the form

$$|V>= exp(\sum_{i,j}\sum_{n,m} a^{ij}_{nm}\psi^i_{-n-1/2}\psi^{*i}_{-m-1/2})|0> \quad (27)$$

where the point of Grassmanian representing the topological vertex was derived in [23].

Hence we can define the classical τ function of the 3-KP system we are working with

$$\tau(T_k) =< t|\Psi(z_1)....\Psi(z_k)|V_{tv}> \quad (28)$$

It is this tau-function of the 3-KP system that plays the role of the generating function for the MHV amplitudes. In fact the semiclassical limit of the tau-function is of the most interest when we consider the classical Riemann surface before any quantization. In the semiclassical approximation we can safely consider the differential

$$dS = vdu \quad (29)$$

which yields the semiclassical brane wave function

$$\Psi_{qs} \propto exp(-\hbar^{-1}\int^x v(u)du) \quad (30)$$

involving the dilogs. The tau-function obeys the 3-KP equation and there are the additional $W_{1+\infty}$ Ward identity written in terms of the fermions

$$\oint_u \psi^*(u)e^{nu}\psi(u) + (-1)^n \oint_v \psi^*(v)e^{nv}\psi(v) + \oint_s \psi^*(s)e^{ns}\psi(s) = 0 \quad (31)$$

where the sum over three asymptotic regions is considered.

The quantization of the system can be done most effectively in terms of the Baxter equation. The Baxter equation implies that the regulator branes are localized on the surface. Hence the whole set of the equations determining amplitudes involve the dual conformal transformations on the regulator worldvolume and the set of Ward identities for the coordinate of regulator brane in the transverse moduli space. It is these Ward identities which fix the dependence of the amplitude on the conformal invariants for large number of external legs.

The precise higher Hamiltonians from $W_{1+\infty}$ responsible for the higher conservation laws in the scattering amplitude problem can be written as the fermionic bilinears [23]. Generically as was discussed in [23] one has some unbroken part of W_∞ which annulate the τ-function corresponding to the topological vertex and therefore the scattering amplitude in the form of BDS anzatz.

On the Faddeev-Volkov model

Let us turn now to the description of the second integrable system representing the particular solitonic sector of the infinite-dimensional integrable system. We shall conjecture that the integrable system at the generic kinematics is the generalization of the SL(2,C) spin chain relevant for the Regge limit of the amplitudes.

The finite-dimensional integrable systems can be usually defined in terms of the R-matrix. The Faddeev-Volkov model is defined via the Drinfeld solution to the Yang-Baxter equation which provides the universal R-matrix acting on $U_q(SL(2,R))\otimes U_q(SL(2,R))$. The corresponding statistical model describes the discrete quantum Liouville theory [31] with the following partition function

$$Z=\int\prod_{ij}W_{p-q}(S_i-S_j)\prod_{kl}\bar{W}_{p-q}(S_k-S_l)\prod_i dS_i \quad (32)$$

where the Boltzmann weights depend only on the differences of the spins S_k at the neighbor cites and rapidity variables at the ends of the edge. The first product is over the horizontal edges(i,j) while the second product is over the vertical edges (k,l). The integral is over all internal spin degrees of freedom. In the fundamental R-matrix the cross-ratios of the relative rapidities of the particles play the role of the local inhomogeneities in the lattice model and Boltzmann weights are defined as [31]

$$W_\theta(s)=F(\theta)^{-1}e^{2\eta\theta s}\frac{\Psi(s+ic_b\theta/\pi)}{\Psi(s-ic_b\theta/\pi)} \quad (33)$$

where spin s and local rapidity variables θ are combined together in the argument of the function

$$\Psi_b(z)=exp(\frac{1}{4}\int\frac{e^{-2izx}dx}{xsinh(bx)sinh(b^{-1}x)}) \quad (34)$$

$c_b=1/2(b+b^{-1})$ and $F(\theta)$ is normalization factor. The relative importance of the spin variables and the local inhomogeneities depends on the value of the YM coupling constant and the kinematical region.

Semiclassically when $b\to 0$ the spin variables are frozen and the Boltzmann weight behaves as

$$W_\theta(\rho/2\pi b))=exp(-\frac{A(\theta|\rho)}{2\pi b^2}+...) \quad (35)$$

where

$$A(\theta|\rho)=iLi_2(-e^{\rho-i\theta})-iLi_2(-e^{\rho+i\theta}) \quad (36)$$

The extremization of the semiclassical action yields the Bethe Anzatz type equations connecting the dynamical spin variables with the local rapidities

$$\prod_i\frac{e^{\rho_i}+e^{\rho_j+\theta_{ij}}}{e^{\rho_j}+e^{\rho_i+\theta_{ij}}}=1 \quad (37)$$

The Regge limit is described in terms of the SL(2,C) spin chains when the number of sites in the chain corresponds to the number of reggeons. The possible limit which could yield such spin chain from the Faddeev-Volkov model or statistical model [31] looks as follows. In the model [31] the statistical weights depend on the sum of the local rapidities and the spin variables. It is clear that one can not expect the quasiclassical limit of the quantum dilogarithm to be relevant since the reggeization of the gluon happens upon the nontrivial resummation of the perturbation series.

Fortunately there is the limit [31] corresponding to the strong coupling region in the Liouville theory when the quantum dilogarithms reduce to the ratio of Gamma functions depending on the $SL(2,R)$ spin variables

$$\Psi_{c_b\to 0}(s+\eta x)\propto\frac{\Gamma(1-s+ix/2)}{\Gamma(1-s-ix/2)} \quad (38)$$

where $|b|=1$. The leading argument depends on the difference of two infinite-dimensional representations in the neighbor sites and the expression coincides with the fundamental R-matrix involved into the $SL(2,R)$ spin chains. That is in this particular limit we get the statistical weights or R-matrixes depending only on the SL(2,R) spins similar to the BFKL-type Hamiltonian [47] while the local rapidity yields the "time" variable $logs$. Note that clearly this suggestive argument need for further clarification.

CONCLUSION

We have suggested the relation between the loop MHV amplitudes and the KS gravity in the momentum space which allows us to recover the relevant integrability pattern. The key idea is that the scattering of the particles induces the back-reaction on the geometry of the momentum space through the nontrivial dynamics on the emerging moduli space. That is one can say that the tree amplitude is dressed by the gravitational degrees of freedom which can be treated within the Kahler gravity in the A type geometry or KS gravity in the type B model. They are identified with the coordinates of Lagrangian branes in the A model or the corresponding noncompact branes in the B model. On the field theory side the four-fermion correlator on the moduli space is identified with the two-mass easy box amplitude which is the basic block in the whole answer.

The BDS anzatz corresponds to the semiclassical limit in the KS gravity and Γ_{cusp} has to be identified with the inverse Planck constant in KS gravity. There are several natural generalizations of the BDS anzatz. First one could imagine that the quantization parameter can be generalized to more complicated function than cusp anomalous dimension respecting the S-duality of N=4 theory. The next evident point concerns the full quantum theory in the KS framework which effectively substitutes the dilogarithm function in the BDS anzatz by the quantum dilogarithm. However these modifications do not produce higher polylogaritms which are known to appear in higher loop calculations of the amplitudes and Wilson polygons. The most natural way to get higher polylogarithms in our picture is to consider the nontrivial Feynman diagrams in the two-dimensional KS theory probably involving loops. Indeed increasing the number of vertexes in the KS tree diagrams we increase the trancendentality of the answer. We expect that all mentioned generalizations are necessary to get the correct all-loop answer.

We have identified the most natural integrable structure behind the scattering amplitudes which are considered as a kind of the "wave functions" in the particular model. The KS gravity in our case naturally involves the 3-KP hierarchy and the role of the "time" variables are played by the combination of the conformal cross-ratios. The second finite-dimensional integrable system is conjectured to be related to the Faddeev-Volkov model however this point deserves for further investigation. The integrability is responsible for the conservation laws in addition to the dual superconformal symmetry. The relevant Ward identities correspond to the area preserving symplectomorphysms of the spectral curve.

The additional IR regulator branes added into the picture are responsible for the blow up of the internal momentum space in the manner dictated by the scattering process. The blow up of the internal geometry physically corresponds to the IR regularization of the field theory and the anomaly in the transformations in the momentum space tells that the regularization does not decouple completely. This a little bit surprising picture implies that we have to take into account the dynamics of the regulator degrees of freedom as well. Naively they are treated semiclassically but generically the fermions representing the regulator branes obey the quantum Baxter equation.

One of the most inspiring findings is the appearance of the hidden "new massive degree of freedom". They correspond on the A model side to the D2 brane wrapped around the 2-cycle created by the scattering states or the open string stretched between two IR regulator branes in the B model. It is somewhat similar to the W-boson state however its mass is fixed by the kinematical invariants of the scattering particles. In the Regge limit we anticipate its important role in the Reggeon field theory.

In is evident that the results of this paper are qualitative in many respects and represent only part of the whole picture. In particular the clear understanding of the amplitudes of the gluon scattering with generic chiralities is absent and our proposal for the improvement of the BDS anzatz deserves for the further evidences. Nevertheless we believe that the dual picture we have suggested is the useful step towards the clarification of the scattering geometry responsible for the summation of the perturbative series in YM theory.

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