Quantized Black Holes and Their Radiation

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Abstract. Temperature and entropy of black holes are discussed. The maximum entropy of a quantized surface is demonstrated to be proportional to the surface area in the classical limit. The general structure of the quantum spectrum of a black hole horizon is found. The discrete spectrum of thermal radiation of a black hole fits the Wien profile. The natural widths of the lines are much smaller than their separation.

Keywords: Black holes; entropy; adiabatic invariance; discrete spectrum
PACS: 97.60.-s

TEMPERATURE OF BLACK HOLES

Long ago, in 1971, J. Wheeler realized that the classical description of black holes is incomplete in principle. His line of reasoning looked as follows. Let us take a box filled with the black body radiation at some temperature T. Obviously it possesses a finite entropy as well. We drop the box into a black hole. Then the entropy of the observable part of the universe decreases forever. But this is an explicit violation of the second law of thermodynamics! To save the second law, Bekenstein suggested [1] that the black hole itself has some entropy which increases when the box is absorbed. Then it is only natural to ascribe some finite temperature as well to a system with a finite entropy. This conclusion is quite natural from a somewhat different point of view. A black hole is an ideal absorber, an absolutely black body, for which the temperature is a quite natural property.

Let us try at first to estimate this temperature just by dimensional arguments. The classical parameters at our disposal, the Newton gravitational constant k, the mass M of black hole, and the speed of light c, are insufficient for the purpose (Mc^2 is too large, and does not contain k). But there is the Planck constant $\hbar$. With it one can easily construct the necessary combination: the black hole temperature is on the order of magnitude $\hbar c^3/(kM)$.

To derive the numerical factor in the relation $T \sim \hbar c^3/(kM)$, we consider the following problem [2]. Let a semiclassical wave packet of a massless field propagate from a point $r_0 = r_g + \varepsilon$ close to the horizon of a black hole with mass M ($r_g = 2kM/c^2$ is the horizon radius) to a distant point r. A straightforward (but rather tedious) calculation demonstrates that, independently of the initial spectrum of the wave packet, at infinity it is completely universal:

$$|f(\omega)|^2 \sim \exp\left(-8\pi kM\omega/c^3\right). \tag{1}$$

If one goes over in it from the frequency ω to the energy $\hbar\omega$, it corresponds to the Boltzmann distribution with the temperature

$$T = \hbar c/(4\pi r_g) = \hbar c^3/(8\pi kM). \tag{2}$$

This expression for the black hole temperature was obtained by S. Hawking [3].

RADIATION OF BLACK HOLES

The inevitable result of the finite temperature T of a black hole is the conclusion that in fact it radiates. Black hole produces not only photons and neutrinos with energies on the order of T, but particles of non-vanishing rest mass m as well (only if its temperature is sufficiently high). Thus, one of the most amazing properties of black holes is that they shine!

V.N. Gribov was the first who made this conclusion[1]. His argument was as follows. The uncertainty relation $\Delta E \Delta t \geq \hbar$ allows the creation of pairs of particles from vacuum for the time t that does not exceed $\hbar/E$; here E is the total energy of the pair ($E \geq 2mc^2$ for massive particles). The gravitational field near the horizon is very strong, so that the energy conservation by itself allows one of the particles to be absorbed by the black hole, and the second one to go to infinity. In quantum mechanics, due to the tunneling effect of such a sort, the processes of particle creation become possible. One can recall in this connection the creation of electron-positron pairs in strong electric fields.

Contrary to naïve expectations, the radiation of black holes is not described by the common Planck and Fermi

[1] Gribov precisely formulated the statement that black holes radiate in discussions taking place in 1971 or 1972. This was told to me independently by A.D. Dolgov, D.I. Diakonov, L.B. Okun', who had been present at those discussions. Unfortunately, Gribov did not publish this result, perhaps he considered it self-evident. In 1974 radiation of black holes was predicted independently by S. Hawking [3].

distributions. Indeed, these distributions are valid only if the size of a radiating body is much larger than the typical wave lengths, i.e. in the semiclassical limit. Let us recall the common model of an absolutely black body: an opening in a cavity filled with radiation [4] (we talk here about photons, but these arguments apply to particles of any spin). Obviously, the radiation of wave lengths large as compared to the size of the opening is strongly suppressed. So, the common temperature distributions apply only under the condition $\omega a \gg 1$, where a is the typical size of a radiating body.

For a black hole, however, the typical frequencies are $\omega \sim (4\pi r_g)^{-1}$ (see (1)), and the size of radiating body a is r_g. Thus, here

$$\omega a \sim 1/(4\pi) \ll 1 \,. \tag{3}$$

As a result, the black body radiation differs essentially from the common thermal one. In particular, due to the centrifugal effect, at $\omega r_g \sim 1/(4\pi) \ll 1$, the density of a partial wave of radiated field near the horizon, and therefore the radiation intensity in this wave, falls down rapidly with the growth of the total angular momentum j. The results of numerical solution of wave equations for particles of different spins in the field of a black hole [5] confirm this qualitative conclusion. The total radiation intensities for (ultrarelativistic) electrons ($j_{\min} = 1/2$), photons ($j_{\min} = 1$), and gravitons ($j_{\min} = 2$) are related as 22:9:1. On the other hand, the ratios of the radiation intensities in the waves with $j = j_{\min}$ and $j = j_{\min} + 1$ (corresponding to a single value of the projection j_z, i.e. without the weight $(2j+1)$) are as follows: for electron 26:1, for photon 47:1, for graviton 95:1. It is curious that, according to the same numerical calculations, the total intensity of the photon radiation by a black hole is quantitatively close to the naïve Planck one.

We mean here sufficiently light black holes, with mass $\sim 10^{15} - 10^{16}$ g, and with typical temperature in the interval $10 - 1$ MeV, correspondingly. Electrons and positrons emitted by such black holes can be treated with good accuracy as ultrarelativistic ones. However, for real black holes the temperature (2) is negligibly small: for the mass comparable with that of the Sun it is only about 10^{-7} K.

As to the stars with masses $\sim 10^{15} - 10^{16}$ g, their gravitational field is too weak, they cannot compress to their gravitational radii, they cannot turn into black holes. Such light black holes in principle could arise at the most early stages of the Universe evolution.

But could these mini-holes survive since those times? Could their age approach the Universe life time $\tau \sim 10^{10}$ years, or 10^{17} s? The problem here is the black hole thermal radiation itself. Let us estimate its intensity I by dimensional arguments: divide T by the characteristic time r_g/c:

$$I \sim cT/r_g \sim m_p^4 c^4/(\hbar M^2). \tag{4}$$

We have introduced here the so-called Planck mass

$$m_p = (\hbar c/k)^{1/2} = 2.2 \times 10^{-5}\,\mathrm{g}\,. \tag{5}$$

Obviously, $I = -c^2 dM/dt$. Solving the differential equation

$$dM/dt = -m_p^4 c^2/(\hbar M^2)\,,$$

we find that to survive until our time a black hole should have an initial mass

$$M > m_p\,(\tau/t_p)^{1/3} \sim 10^{15}\,g\,. \tag{6}$$

Here t_p is the so-called Planck time

$$t_p = \hbar/(m_p c^2) = \left(\hbar k/c^5\right)^{1/2} = 0.54 \times 10^{-43}\,\mathrm{s}\,. \tag{7}$$

Together with the energy, a black hole loses its mass. Then, according to relation (4), the intensity of its radiation grows, and the gravitational radius of a black hole gets smaller. However a star cannot radiate more energy than it has. The radiation stops when the black hole temperature becomes comparable to its rest energy, at

$$Mc^2 \sim T \sim m_p^2 c^2/M\,,$$

i.e. when the mass of such a mini-hole decreases to the Planck mass:

$$M \sim m_p\,.$$

Here our semiclassical consideration of quantum effects becomes inapplicable, and a consistent quantum theory of gravity is necessary.

It is instructive to look at relation (2) somewhat otherwise. It demonstrates that the energy of a black hole, together with its mass, decreases as the temperature increases. Thus, the heat capacity of a black hole is negative. This unusual property is quite typical for gravitating systems in general [4]. As to a black hole, its negative heat capacity is directly related to the instability caused by radiation. Let us recall, however, that the classical instability of an electron bound in the Coulomb field, also caused by radiation, is stabilized by quantum effects. In the case of black holes as well, it is natural to assume that on the Planck scale their semiclassical radiative instability is stabilized by quantum effects.

One more fact related to the radiation of black holes. For the typical time interval $\Delta t \sim r_g/c$ between the acts of radiation, the uncertainty of the energy of a black hole is $\Delta E \sim \hbar/\Delta t \sim \hbar c^3/kM$. The corresponding uncertainty in the gravitational radius is [6]

$$\Delta r_g \sim k\Delta M/c^2 \sim k\Delta E/c^4 \sim \hbar/(Mc)\,.$$

Obviously, at least due to this uncertainty, the time of the fall of a point-like particle to the horizon (which is logarithmically divergent in the classical approach!) becomes finite:

$$t \simeq r_g \ln r_g/\Delta r_g \simeq r_g \ln M^2/m_p^2 .$$

The arising logarithm is huge, $\ln M^2/m_p^2 \simeq 10^2$, but here it is not of much importance.

ENTROPY AND HORIZON AREA

Now, when the temperature of a black hole is known, its entropy is calculated with the well-known thermodynamical formula $dE = T\,dS$. In our case T is given by formula (2), and $E = Mc^2$. Solving the differential equation

$$dM = \hbar c\, dS/(8\pi k M)$$

with the natural boundary condition $S = 0$ for $M = 0$, we find $S = 4\pi k M^2/(\hbar c)$. It is convenient to introduce the so-called Planck length

$$l_p = (\hbar k/c^3)^{1/2} = 1.6\times 10^{-33}\ \text{cm}. \tag{8}$$

Then we arrive at the following remarkable relation between the entropy of a Schwarzschild black hole and the area of its horizon $A = 4\pi r_g^2$:

$$S = \pi r_g^2/l_p^2 = A/(4 l_p^2). \tag{9}$$

The corresponding analysis for a charged black hole is more intricate. In the Schwarzschild case, the horizon area $A = 16\pi k^2 M^2$ (from now on, we put $c = 1$) depends on the only parameter M, and the adiabatic invariance of A means that M is also an adiabatic invariant. But the horizon area of a charged black hole depends not only on its mass, but on its charge q as well (see, e.g., [7]):

$$A_{ch} = 4\pi r_{gc}^2; \quad r_{gc} = kM + \sqrt{k^2M^2 - kq^2}. \tag{10}$$

So, what happens with this black hole when a small charge e is lowered adiabatically to its horizon? What remains constant, the horizon area or the mass (if either)?

To answer the question, we resort to a thought experiment. Let a particle with energy ε at infinity and charge e falls radially to the horizon. The charges e and q are of the same sign, so that the electrostatic repulsion could compensate for the gravitational attraction. As a result of the capture, the black hole mass increases by $\Delta M = \varepsilon$, and its charge by $\Delta q = e$ (both energy and charge are conserved). The change of the horizon area in this process is

$$\Delta A_{ch} = \frac{8\pi r_{gc} k}{\sqrt{k^2M^2 - kq^2}} \left(\varepsilon - \frac{eq}{r_{gc}}\right). \tag{11}$$

If one chooses $\varepsilon = eq/r_{gc}$, then for a non-extremal black hole (i.e. for $q^2 < kM^2$) this expression turns to zero. Let us prove now that at the capture the particle velocity tends to zero, i.e. that the capture itself is adiabatic. To this end, we use the identity $g_{\mu\nu}u^\mu u^\nu = 1$. In the present case of diagonal metric and radial motion, this identity reduces to $g_{00}(u^0)^2 + g_{rr}(u^r)^2 = 1$, or

$$g^{00}(u_0)^2 + g_{rr}(u^r)^2 = 1. \tag{12}$$

In this case

$$g^{00} = -g_{rr} = a^{-2}(r) = \left(1 - 2kM/r + kq^2/r^2\right)^{-1}.$$

By definition, $u_0 = (p_0 - eA_0)/m = (\varepsilon - eq/r)/m$. We recall here that the zeroth component of the covariant momentum is the conserved particle energy (it gets obvious in the Hamilton-Jacobi formalism). Again by definition, $(u^r)^2 = (dr/d\tau)^2$, where τ is the invariant, local time. Finally, equation (12) is rewritten as follows:

$$(\varepsilon - eq/r)^2 = m^2\,(dr/d\tau)^2 + m^2a^2. \tag{13}$$

At the horizon, at $r = r_{gc}$, a vanishes. It gets clear now that at the capture by a charged black hole of a particle with energy $\varepsilon = eq/r_{gc}$, its radial velocity measured by a local observer, $dr/d\tau$, tends to zero, and the duration of this process measured by the same clock tends to infinity. Thus, the area A of the horizon of a charged black hole (but not the mass) is the adiabatic invariant[2]. We recall now that the entropy remains constant under adiabatic processes. Therefore, just $A/(4l_p^2)$ should be identified with the entropy of a charged black hole.

QUANTIZATION OF BLACK HOLES. HOLOGRAPHIC BOUND

On the other hand, the quantization of an adiabatic invariant is perfectly natural. And just on this argument was based the idea of quantizing the horizon area of black holes proposed by J. Bekenstein [10].

We start the discussion with eq. (13), rewriting it as

$$\varepsilon - eq/r = \sqrt{m^2\,(dr/d\tau)^2 + m^2a^2}. \tag{14}$$

It is convenient to go over here from the contravariant radial component $p^r = m\,dr/d\tau$ of the momentum vector to the corresponding component π^r in the locally inertial

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[2] Originally, the fact of the adiabatic invariance of the horizon area was established by D. Christodoulu and R. Ruffini for rotating black holes [8, 9]. But here we confine to a more simple case of charged black holes.

frame: $\pi^r = \sqrt{-g_{rr}}\, p^r = p^r/a(r)$. And in the thus arising expression

$$\varepsilon - eq/r = a(r)\sqrt{(\pi^r)^2 + m^2}$$

we go over from r to the coordinate ρ, given also in the locally inertial frame and counted off the horizon:

$$\rho = \int_{r_{gc}}^{r} dr\sqrt{-g_{rr}(r)} = \int_{r_{gc}}^{r} \frac{dr}{\sqrt{a(r)}} = \frac{2\, r_{gc}\sqrt{r - r_{gc}}}{\sqrt{r_{gc} - r_{gc}^{-}}}\,;$$

here $r_{gc}^{-} = kM - \sqrt{k^2M^2 - kq^2}$; we assume that $\rho \ll r_{gc}$. As a result, eq. (14) transforms for $r \to r_{gc}$ to

$$\varepsilon - \frac{eq}{r} = \frac{\sqrt{k^2M^2 - kq^2}}{r_{gc}}\, \rho\, \sqrt{(\pi^r)^2 + m^2}\,. \qquad (15)$$

At last, in virtue of (11), the change of the horizon surface in this process is

$$\Delta A_{ch} = \frac{8\pi k r_{gc}}{\sqrt{k^2M^2 - kq^2}}\left(\varepsilon - \frac{e\,q}{r_{gc}}\right)$$

$$= 8\pi k\sqrt{\rho^2\,(\pi^r)^2 + (m\rho)^2}\,. \qquad (16)$$

Of course, $m\rho$ vanishes in the limit $\rho \to 0$. However, in virtue of the uncertainty relation, $\rho\,\pi^r$ stays finite in this limit: $\rho\,\pi^r \gtrsim \hbar$. Thus, the minimum change of the horizon surface is

$$\Delta A_{min} \simeq 8\pi\hbar k = 8\pi\, l_p^2\,. \qquad (17)$$

Obviously, the fact that the minimum possible change of the horizon area is finite, makes the horizon quantization quite natural. We assume therefore that the whole horizon area A consists of patches of typical size $\sim 8\pi\, l_p^2$. Each of them is characterized by a quantum number j, such that the contribution a of a patch to the area depends on this quantum number, $a = a(j)$. Besides, a patch can possess a quantum number m, such that a is independent of it[3]. Then, the horizon area is conveniently rewritten as

$$A = 8\pi\gamma l_p^2 \sum_{jm} a(j)\,\nu_{jm}\,, \qquad (18)$$

where ν_{jm} is the number of patches of given j and m. The numerical factor γ will be determined below for given function $a(j)$ and given statistical weight $g(j)$ (as usual, the latter equals the number of possible values of m for given j). Correspondingly, the entropy of a black hole is

$$S = 2\pi\gamma \sum_{jm} a(j)\,\nu_{jm}\,. \qquad (19)$$

[3] In principle, both j and m may refer not only to a single quantum number each, but to sets of them: $j = (j_1, j_2, ...)$, $m = (m_1, m_2 ...)$.

The occupation numbers ν_{jm} can be related to $a(j)$ and $g(j)$ by using the so-called holographic bound. According to it, the entropy S of any spherical nonrotating system confined inside a sphere of area A is bounded by relation

$$S \le A/(4l_p^2)\,, \qquad (20)$$

with the equality attained only if the system is a black hole [11–13].

A simple intuitive argument confirming this bound is as follows [13]. Let us allow the discussed system to collapse into a black hole. Due to the spherical symmetry, this process is not accompanied by radiation or any other loss of matter. During the collapse the entropy increases from S to S_{bh}, or at least remains constant. And the resulting horizon area A_{bh} is certainly smaller than the initial confining one A. Now, with the account for relation (9) for a black hole, we arrive, through the obvious chain of (in)equalities

$$S \le S_{bh} = A_{bh}/(4l_p^2) \le A/(4l_p^2)\,,$$

at the discussed bound (20).

The holographic bound looks quite surprising since usually the entropy of a body is proportional to its volume, but not to the area of its surface. However, in regular situations limit (20) is so weak quantitatively that no contradiction with the common sense arises. In fact, at least for spherically symmetric black holes, the holographic bound has been checked by careful analysis of various physical situations, and therefore its validity is firmly established.

The result (20) can be formulated otherwise. Among the spherical surfaces of a given area, it is the surface of a black hole horizon that has the largest entropy.

Let us come back now to our problem. We will consider the "microcanonical" entropy S of a quantized surface, defined as the logarithm of the number of states of this surface for a fixed value A of its area (instead of fixed energy in common problems).

Obviously, this number of states K depends essentially on the assumption concerning the distinguishability of the patches. So, let us discuss first of all which of a priori possible assumptions is reasonable here from the physical point of view [14].

We start with the possibility of complete indistinguishability of patches. It means that for given ν_{jm} no permutation of any patches results in new states, i.e. this is the only state at all. Correspondingly, the entropy in this case just turns to zero.

Let us consider now the opposite assumption, that of completely distinguishable patches. In this case the total number of states is

$$K = \nu!\,, \quad \nu = \sum_j \nu_j = \sum_{jm} \nu_{jm}\,,$$

with the microcanonical entropy[4]

$$S = \nu \ln \nu \,.$$

Obviously, here the maximum entropy for fixed $A \sim \sum_j a(j)\ \nu_j$ is attained with all $a(j)$ being as small as possible. Then, in the classical limit $\nu \gg 1$, the entropy of a black hole grows faster than its area: $A \sim \nu$, but $S = \nu \ln \nu \sim A \ln A$. Thus, the assumption of complete distinguishability is in conflict with the holographic bound, and therefore should be discarded.

Now the third possibility (used to be quite popular). Here the total number of states and the entropy are

$$K = \prod_j g(j)^{\nu_j}\,, \quad \text{and} \quad S = \sum_j \nu_j \ln g(j). \tag{21}$$

This scheme corresponds in fact to the following assumptions on the distinguishability of patches:

nonequal j,	any m	$\longrightarrow$	indistinguishable;
equal j,	nonequal m	$\longrightarrow$	distinguishable;
equal j,	equal m	$\longrightarrow$	indistinguishable.

The combination of the first two of them looks strange and unnatural (except the special case when only a single value of j is allowed for all patches).

The only reasonable set of assumptions on the distinguishability of patches, which may result in acceptable physical predictions (i.e. may comply both with the relation (9) between the entropy and the horizon surface, and with the holographic bound (20)) is as follows:

nonequal j,	any m	$\longrightarrow$	distinguishable;
equal j,	nonequal m	$\longrightarrow$	distinguishable;
equal j,	equal m	$\longrightarrow$	indistinguishable.

Under these assumptions, the number of states of the horizon surface, for a given number ν_{jm} of patches with quantum numbers j and m, is obviously [21]

$$K = \nu\,! \prod_{jm} \frac{1}{\nu_{jm}\,!}\,, \text{ where } \nu = \sum_j \nu_j\,,\ \nu_j = \sum_m \nu_{jm}\,, \tag{22}$$

and the corresponding entropy equals

$$S = \ln K = \ln(\nu\,!) - \sum_{jm} \ln(\nu_{jm}\,!)\,. \tag{23}$$

The structures of the last expression and of formula (19) are so different that in a general case the entropy certainly cannot be proportional to the area. However, this is the case for the maximum entropy. We will calculate it for a fixed area, i.e. for a fixed sum

$$N = \sum_{jm}^{\infty} a(j)\,\nu_{jm} = \text{const}. \tag{24}$$

The problem reduces to the solution of the system of equations

$$\ln \nu - \ln \nu_{jm} = \mu\, a(j)\,, \tag{25}$$

where μ is the Lagrange multiplier for the constraining relation (24). These equations can be rewritten as

$$\nu_{jm} = \nu\, e^{-\mu\, a(j)}, \quad \text{or} \quad \nu_j = \nu\, g(j)\, e^{-\mu\, a(j)}. \tag{26}$$

Now we sum expression (26) over j, and with $\sum_j \nu_j = \nu$ arrive at the equation for μ:

$$\sum_j g(j)\, e^{-\mu\, a(j)} = 1. \tag{27}$$

On the other hand, multiplying equation (25) by ν_{jm} and summing over jm, we arrive, with the constraint (24), at the following result for the maximum entropy for given N:

$$S_{\max} = \mu N = \mu A/(8\pi\gamma l_p^2)\,. \tag{28}$$

Thus, equation (18) for the quantized area can be written as

$$A = 8\pi\gamma l_p^2\, \nu \sum_j g(j)\, a(j)\, e^{-\mu a(j)}\,, \tag{29}$$

where $\gamma = \mu/(2\pi)$, and the value of μ is found from equation (27).

Let us note that, strictly speaking, the summation in formulae (27), (29) goes not to infinity, but to some j, corresponding to the maximum contribution $a_{\max}$ to the horizon area. The value of $a_{\max}$ follows from the obvious condition: none of the occupation numbers ν_{jm} should be less than unity. Then equation (26) results in the estimate

$$a_{\max} \sim \ln \nu/\mu\,. \tag{30}$$

We illustrate now these general relations with an example of a concrete model, that of loop quantum gravity (LQG) [16]. A quantized surface in LQG looks as follows. One ascribes to it a set of punctures (corresponding to our patches). Each puncture is supplied with an integer or half-integer quantum number j:

$$j = 1/2, 1, 3/2, \ldots\,. \tag{31}$$

The projections m of these "angular momenta" (unrelated to the common ones) run as usual from $-j$ to j. The area of a surface is

$$A = 8\pi\gamma l_p^2 \sum_{jm} \sqrt{j(j+1)}\,\nu_{jm}. \tag{32}$$

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[4] We assume that all occupation numbers are sufficiently large, so that the simple Stirling approximation is applicable for all factorials.

This is in fact a special case of the above general expressions with

$$a(j) = \sqrt{j(j+1)}, \quad g(j) = 2j+1. \tag{33}$$

The numerical factor γ in (32) (the so-called Barbero–Immirzi parameter) corresponds in LQG to a family of nonequivalent quantum theories, all of them being a priori, without additional arguments, viable [17, 18]. In this case, our "secular" equation (27) and its solution are, respectively, [15, 19]:

$$\sum_{j=1/2}^{\infty}(2j+1)\, e^{-\mu\sqrt{j(j+1)}} = 1, \;\; \gamma = \frac{\mu}{2\pi} = 0.274. \tag{34}$$

QUANTIZATION OF ROTATING BLACK HOLES

In the next section we will discuss the radiation spectrum of quantized black holes. Here, generally speaking, one should take into account the selection rules for angular momentum. Therefore, the quantization rule for the mass of a Schwarzschild black hole should be generalized to that of a rotating Kerr black hole.

To derive the quantization rule for Kerr black hole, we come back to the thought experiment analyzed in [8, 9]. Therein, under the adiabatic capture of a particle with an angular momentum j, the angular momentum J of a rotating black hole changes by a finite amount j, but the horizon area A does not change. Of course, under some other variation of parameters it is the angular momentum J that remains constant. In other words, we have here two independent adiabatic invariants, A and J, for a Kerr black hole with a mass M.

Such a situation is quite common in ordinary mechanics. For instance, the energy of a particle with mass m, bound in the Coulomb field $U(r) = -\alpha/r$, is

$$E = -\frac{m\alpha^2}{2\,(I_r + I_\phi)^2}\,, \tag{35}$$

where I_r and I_ϕ are adiabatic invariants for the radial and angular degree of freedom, respectively. Of course, the energy E is in a sense an adiabatic invariant also, but it is invariant only with respect to those variations of parameters under which both I_r and I_ϕ remain constant (or at least their sum). As to quantum mechanics, in it formula (35) goes over into

$$E = -\frac{m\alpha^2}{2\hbar^2\,(n_r + 1 + l)^2}\,, \tag{36}$$

where n_r and l are the radial and orbital quantum numbers, respectively.

This example prompts the solution of the quantization problem for a Kerr black hole. It is conveniently formulated in terms of the so-called irreducible mass M_{ir} of a black hole, related by definition to its horizon radius r_h and area A as follows:

$$r_h = 2kM_{ir}\,, \quad A = 16\pi k^2 M_{ir}^2\,. \tag{37}$$

Together with the horizon area A, the irreducible mass is an adiabatic invariant. In accordance with (18) and (24), it is quantized as follows:

$$M_{ir}^2 = \frac{1}{2}\, m_p^2 \gamma N\,, \tag{38}$$

where $m_p^2 = \hbar c/k$ is the Planck mass squared.

Of course, for a Schwarzschild black hole M_{ir} coincides with its ordinary mass M. However, for a Kerr black hole the situation is more interesting. Here [9]

$$M^2 = M_{ir}^2 + \frac{J^2}{r_h^2} = M_{ir}^2 + \frac{J^2}{4k^2 M_{ir}^2}\,, \tag{39}$$

where J is the internal angular momentum of a rotating black hole. Let us note that equation (39) describes a relativistic rotator with the rest mass M_{ir} and the moment of inertia $M_{ir} r_h^2$ (defined as usual in the nonrelativistic limit) [7].

Now, with equation (38), we arrive at the following quantization rule for the mass squared M^2 of a rotating black hole:

$$M^2 = \frac{1}{2}\, m_p^2 \left[\gamma N + \frac{J(J+1)}{\gamma N}\right]. \tag{40}$$

Obviously, as long as a black hole is far away from an extremal one, i.e. while $\gamma N \gg J$, one can neglect the dependence of M^2 on J.

RADIATION SPECTRUM OF QUANTIZED BLACK HOLE

It follows from expression (40) that for a rotating black hole the radiation frequency ω, which coincides with the loss ΔM of the black hole mass, is

$$\omega = \Delta M = T\mu\,\Delta N + \frac{1}{2kM}\,\frac{J+1/2}{\gamma N}\,\Delta J = T\Delta S + \Omega\,\Delta J\,, \tag{41}$$

where ΔN and ΔJ are the losses of the area quantum number N and of the angular momentum J, respectively; $\Delta S = \mu\,\Delta N$, and the effective angular velocity is $\Omega = (J+1/2)/(Mr_h^2)$. Obviously, equation (41) is in fact a common thermodynamic relation.

We will be interested mainly in the first, temperature term in (41), dominating for black holes far from the extremal regime, i.e. for $J \ll \gamma N$. Just this effect is discussed in detail below.

As to the non-temperature radiation of a black hole close to an extremal one, as described by the term with ΔJ in (41), this effect is due to the tunnelling (see relatively recent discussion of this problem in [20, 21]). Loss of the charge by a charged extremal black hole occurs due to the Coulomb repulsion between the black hole and emitted particles of the same sign of the charge. For an extremal rotating black hole, the radiation is caused by the interaction of angular momenta: particles (massless mainly), with the total angular momenta parallel to that of the black hole, are repelled from it.

But let as come back to the temperature radiation. The natural assumption is that it occurs when a patch with a given value of j disappears, which means that

$$\Delta N_j = a(j)\,, \quad \omega_j = T\mu\, a(j)\,. \tag{42}$$

Thus we arrive at the discrete spectrum with a finite number of lines. Their frequencies start at $\omega_{\min} = T\mu\, a_{\min}$, where $a_{\min}$ is the minimum value of $a(j)$, and terminate at $\omega_{\max} \sim T \ln \nu$ (we recall here that $a_{\max} \sim \ln \nu/\mu$). Thus, the number of lines is not so large, $\sim 10^2$, if the mass of black hole is comparable to that of the Sun. However, due to the exponential decrease of the radiation intensity with ω (see below), the existence of $\omega_{\max}$ and finite number of lines are not of much importance.

To substantiate the made assumption, we come back to the lower bound (17) on the change of the horizon area under an adiabatic capture of a particle. The presence of the gap (17) in this process means that this threshold capture effectively consists in the increase by unity of a single occupation number ν_{jm}. If the capture were accompanied by a change of few occupation numbers, some of them increasing and some of them decreasing, the change of the area could be made in general as small as one wishes [5].

It is only natural to assume that in the radiation process as well, the change of few occupation numbers, instead of one, is at least strongly suppressed. In this way we arrive at equations (42).

Our next assumption, at least as natural as this one, is that the probability of radiation of a quantum with frequency ω_j is proportional to the occupation number ν_j. Correspondingly, the radiation intensity I_j at this frequency ω_j is proportional to $\nu_j\, \omega_j$:

$$I_j \sim \nu_j\, \omega_j \sim \nu\, g(j)\, \omega_j\, e^{-\omega_j/T}\,. \tag{43}$$

Thus, we have arrived in a natural way at the exponential Wien profile for $\omega_j \gg T$. The conclusion that the discrete thermal radiation spectrum of a black hole should fit the Wien profile was made in [22] for the case of equidistant horizon quantization

Numerical estimates demonstrate that the total radiation intensity of photons by quantized black holes is about the same as that of classical ones (of course, if $\mu a_{\min} \lesssim 1$). It follows also from the same estimates that the total natural widths of the radiation lines of quantized black holes do not exceed few percent of the line separations. Thus, the radiation spectrum is really discrete.

[5] Except the case when $a(j)$ is a linear function of j, and, correspondingly, the area spectrum is equidistant. Generally speaking, this case cannot be excluded.

ACKNOWLEDGMENTS

I am grateful to A.A. Pomeransky for useful discussions. The work was supported by the Russian Foundation for Basic Research through Grant No. 08-02-00960-a.

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