Entanglement in Quantum Field Theory*

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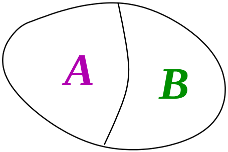
Outline

- entanglement entropy as a measure of quantum entanglement
- path integral approach to entanglement entropy in QFT
- universal results in 1+1 dimensions:
  - conformal field theory
  - form factor approach in massive QFT
- higher dimensions

A. Quantum Entanglement (Bipartite)

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



- A can make observations only in *A*, B in the complement *B*
- in general A's measurements are entangled with those of B

Measuring entanglement

- Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j \, |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$.

- one measure of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

- if $c_1 = 1$, rest zero, $S = 0$ and $|\Psi\rangle$ is unentangled
- if all c_j equal, $S \sim \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$ – maximal entanglement

- equivalently, in terms of A's reduced density matrix:

$$\rho_A \equiv \text{Tr}_B \, |\Psi\rangle\langle\Psi|$$

entropy:

$$S_A = -\text{Tr}_A \, \rho_A \log \rho_A$$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, ...
- in quantum information theory, it gives the efficiency of conversion of partially entangled $\rightarrow$ maximally entangled states by local operations (Bennet et al)
- it gives the amount of classical information required to specify ρ_A (important for density-matrix RG)

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- it gives a basis-independent way of identifying and characterising quantum phase transitions

In this talk we consider the case when:

- the degrees of freedom of the quantum system are extended over some large region $\mathcal{R}$ in $\mathbb{R}^d$
- the hamiltonian H contains only short-range interactions, e.g:
  - a lattice quantum spin system
  - a UV cut-off quantum field theory
- A is the set of degrees of freedom in some large (compact) subset of $\mathcal{R}$
- the whole system is in a pure state, usually the ground state $|0\rangle$ of H (although it will be useful to consider this as the limit $\beta \to \infty$ of a thermal mixed state with $\rho \propto e^{-\beta H}$)

- since first papers (Osborne and Nielsen 2002; Osterloh *et al* 2002; Vidal *et al* 2003), $>$ 150 papers, mostly on exact or numerical calculations in quantum spin models [Review: (Amico *et al* 2007)]

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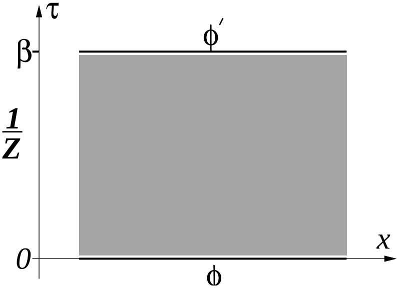
How does S_A depend on the size and geometry of A and the universality class of the critical behaviour?

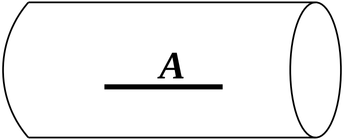
Entanglement entropy from the path integral

$\rho(\phi',\phi) = \frac{1}{Z}$



$Z =$

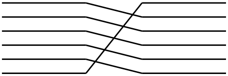
Reduced density matrix

$\rho_A = \frac{1}{Z}$ 

- – sew together the edges along $\tau = 0, \beta$, leaving slit(s) open along A
- now use ‘replica trick’

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A = -\left.\frac{\partial}{\partial n}\right|_{n=1} \mathrm{Tr}\,\rho_A^n$$

- for integer $n \geq 1$, $\mathrm{Tr}\rho_A^n$ is given by n copies of the path integral for S_A, sewn together cyclically along the slits: an n-sheeted Riemann surface



Conformal field theory approach

- suppose the QFT is a 1+1-dimensional CFT (i.e. with a dispersion relation $\omega \sim |k|$)
- uniformising transformation for the n-sheeted surface $\mathcal{R}_n$

$$z = \left(\frac{w - x_1}{w - x_2}\right)^{1/n}$$

- stress tensor

$$\langle T(w) \rangle_{\mathcal{R}_n} = \frac{c}{12}\{z, w\} = \frac{(c/12)(1 - 1/n^2)(x_2 - x_1)^2}{(w - x_1)^2(w - x_2)^2}$$

$$\frac{\partial \log Z_n}{\partial |x_2 - x_1|} = \frac{n}{2\pi}\int_{-\infty}^{\infty} \langle T_{xx}(x_0, t) \rangle dt = -\frac{(c/6)(n - 1/n)}{|x_2 - x_1|}$$

so that $Z_n \sim |x_2 - x_1|^{-(c/6)(n-1/n)}$.

Taking the derivative with respect to n at $n = 1$ we get the final result (Holzhey *et al* 1994)

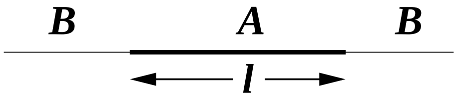
$$S_A \sim (c/3) \log |x_2 - x_1|$$

- in $d = 1$, S grows only logarithmically, even at a critical point: responsible for success of DMRG
- logarithmic behaviour not restricted to $z = 1$ critical points but also eg in random quantum spin chains (Refael and Moore)

- many more universal results, eg finite-temperature cross-over between entanglement and thermodynamic entropy (Korepin 2004, Calabrese + JC 2004):

$$S_A \sim (c/3)\log\big((\beta/\pi)\sinh(\pi|x_2 - x_1|/\beta)\big)$$

Finite correlation length in $d = 1$



- entanglement occurs only over a distance $\sim \xi$ of the contact points between A and B

$$S_\ell \sim 2 \times (c/6) \log \xi$$

with universal corrections $O(e^{-2\ell/\xi})$
(JC,Castro-Alvaredo,Doyon 2007)

Form factor approach

- instead of QFT on n-sheeted surface, consider n copies of the theory in $\mathbf{R}^2$ with fields $(\Psi_1, \ldots, \Psi_n)$ and twist operators $\mathcal{T}_n$ at the location of the branch points, so that

$$
\begin{aligned}
\Psi_i(y)\mathcal{T}_n(x) &= \mathcal{T}_n(x)\Psi_{i+1}(y) \quad (\mathrm{mod}\, n) \qquad (x > y) \\
\Psi_i(y)\mathcal{T}_n(x) &= \mathcal{T}_n(x)\Psi_i(y) \qquad (x < y)
\end{aligned}
$$

- the S-matrix is

$$
\begin{aligned}
S_{ii}(\theta) &= S(\theta) \\
S_{ij}(\theta) &= 1 \qquad\qquad (i \neq j)
\end{aligned}
$$

- define form factors

$$F_{i_1, i_2 \ldots}(\theta_1, \theta_2, \ldots) \equiv \langle 0|\mathcal{T}_n|i_1, \theta_1; i_2, \theta_2; \ldots\rangle_{\mathrm{in}}$$

Form factor equations (2 particles for simplicity)

$$\begin{aligned} F_{ij}(\theta) &= S_{ij}(\theta) F_{ji}(-\theta) \\ F_{ij}(\theta + 2\pi\mathrm{i}) &= F_{j,i+1}(-\theta) \\ \mathrm{Res}_{\theta=\mathrm{i}\pi} F_{i+1,i}(\theta) &= \langle \mathcal{T}_n \rangle \qquad \text{kinematic pole} \end{aligned}$$

- all the F_{ij} are simply obtained from F_{11}:

$$F_{ij}(\theta) = F_{11}(2\pi(j-i)\mathrm{i} - \theta) \qquad (j > i)$$

$$F_{11}(\theta) = F_{11}(2\pi n\mathrm{i} - \theta) = S(\theta) F_{11}(-\theta)$$

- these can be solved in the standard way by first finding a function $F_{11}^{\min}(\theta)$ which has no poles in the region $0 < \mathrm{Im}\,\theta < n\pi$ and then multiplying by factors which insert the correct kinematic poles at $\theta = \pm(2nN - 1)\pi\mathrm{i}$

Two-point function in the two-particle approximation

$$\langle \mathcal{T}(\ell)\mathcal{T}(0)\rangle = \langle \mathcal{T}\rangle^2 + \sum_{i,j=1}^{n} \int \left|F_{ij}(\theta_1-\theta_2)\right|^2 e^{-m\ell(\cosh\theta_1+\cosh\theta_2)} d\theta_1 d\theta_2 + \cdot$$

The sum has the form

$$|F_{11}(\theta)|^2 + \sum_{j=1}^{n-1} |F_{11}(2\pi j\mathrm{i}-\theta)|^2$$

which we have to analytically continue to $n \sim 1$

- this can be done by replacing the sum over j by a contour integral
- the dominant term comes from the kinematic singularities at $j = \frac{1}{2} \pm \frac{\theta}{2\pi\mathrm{i}}$ and $j = n - \frac{1}{2} \mp \frac{\theta}{2\pi\mathrm{i}}$, which pinch the contour as $n \to 1$ and produce a term $\sim (n-1)\delta(\theta)$
- the coefficient of this is universal and independent of the details of $S(\theta)$

Universal result

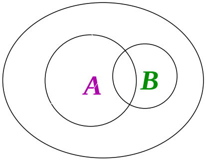
$$S_A(x) \sim \frac{c}{3}\log\xi + U - \frac{1}{8}K_0(2mx) + O(e^{-3mx})$$

- U is also universal and calculable but dependent on the theory ($U_{\mathrm{Ising}} = -0.131984$)
- heuristic arguments (Doyon) show this result should also hold for non-integrable models

Higher dimensions $d > 1$

- in general the leading term in S_A is nonuniversal $\sim a^{1-d}\,\mathrm{Area}(\partial A)$: the area law
- expect a universal term $\sim \xi^{1-d} \times$ area hidden behind this
- however these 'area' terms cancel in

$$S_{A\cup B} + S_{A\cap B} - S_A - S_B$$



- for $z = 1$ quantum critical points this quantity is expected to be universal and given by $c \times$ geometrical factor where c depends on the universality class