Chernogolovka, June 21-26

Statisical physics of dyons and quark confinement

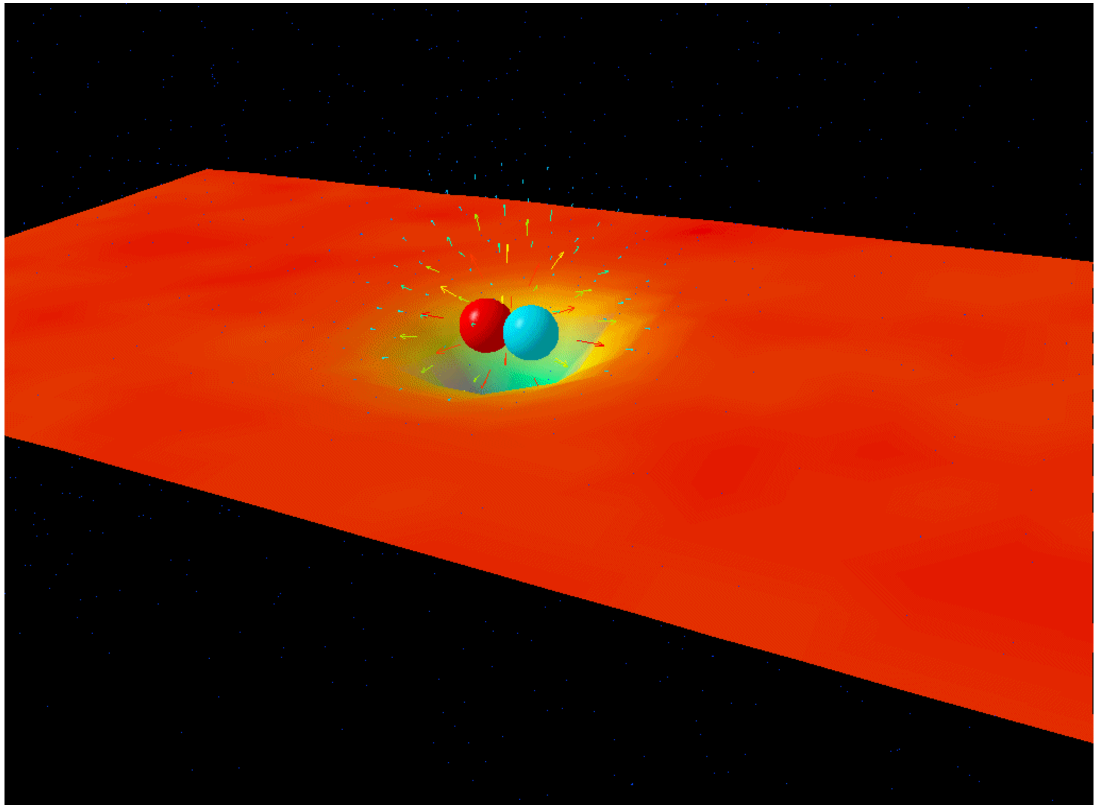
Dmitri Diakonov and Victor Petrov

Petersburg Nuclear Physics Institute

Gatchina

Probably the main problem in particle physics: confinement of colour

All known bound states in physics, **except hadrons**, can be desintegrated into constituents. However, quarks do not exist in a free state. It is assumed that quarks experience a 14 ton attraction, even at an ever increasing separation!

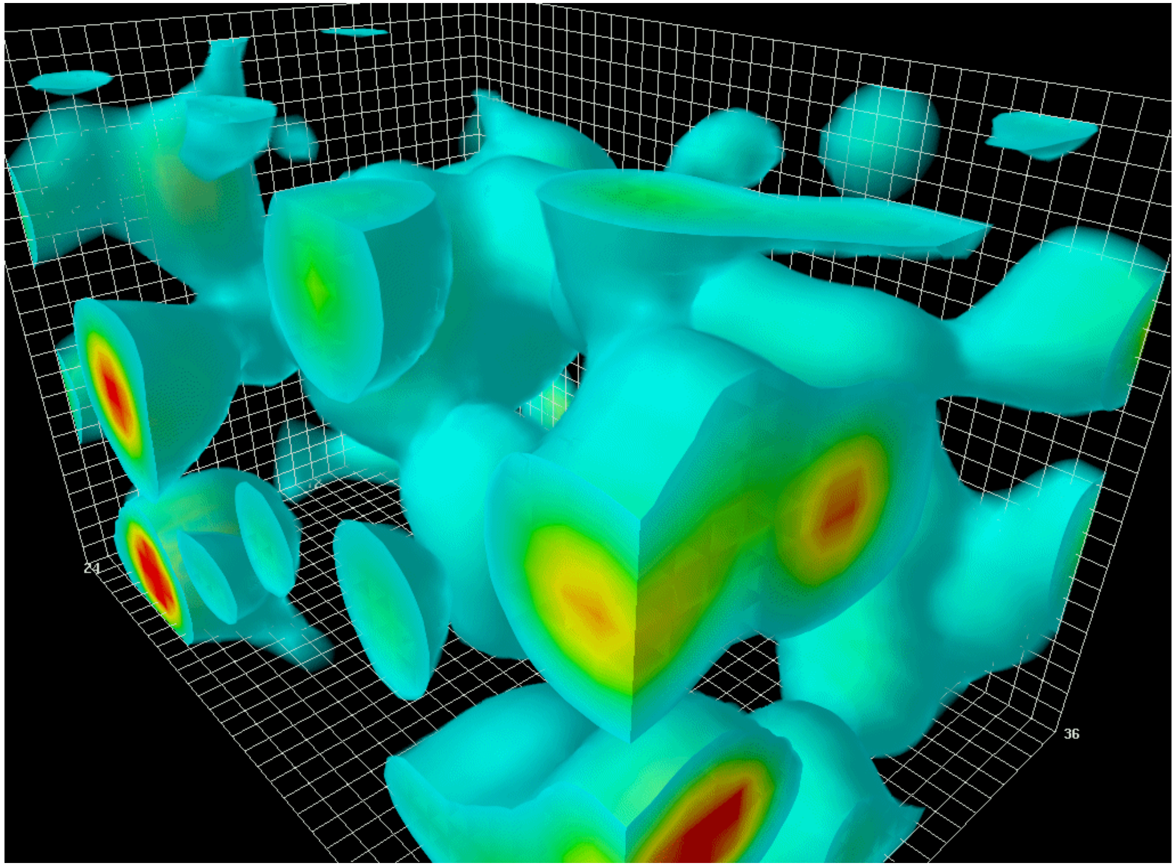


How to explain it, that's the question…

All known forces in Nature decrease with separation…

"Understanding" implies that "14 tons" must be computed!

The empty space (the "vacuum") is populated by large non-linear fluctuations of the Yang – Mills field



The presence of matter doesn't influence vacuum fluctuations too much

One has to guess (better derive) what kind of fluctuations are dominant

Computer simulations of the Yang—Mills vacuum, courtesy Derek Leinweber

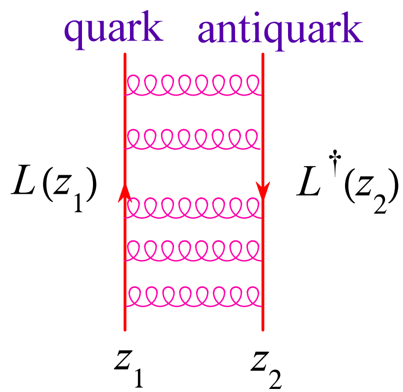
Landau (reportedly): "If there is no small parameter the problem is not formulated". And there seems to be none…

Confinement criteria in a pure Yang – Mills theory (no dynamical quarks):

1) Average Polyakov line

$$< \mathrm{Tr}\ L(\mathbf{z}) > \;=\; \left\langle \mathrm{Tr}\,\mathcal{P}\ \exp\left(i\int_0^{1/T} A_4\,dt \right) \right\rangle$$

$$= e^{-M_{\mathrm{quark}}\,/\,T} \begin{cases} = 0 & \text{below } T_c \\ \neq 0 & \text{above } T_c \end{cases}$$

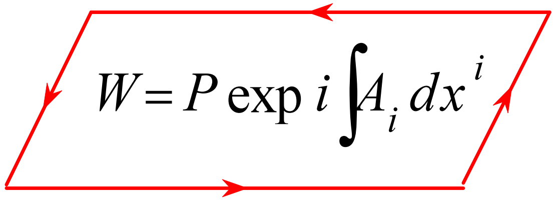


2) Linear rising potential energy of static quark and antiquark

$$< \mathrm{Tr}\ L(\mathbf{z_1})\ \mathrm{Tr}\ L^\dagger(\mathbf{z_2}) >= e^{-V(z_1 - z_2)/T}$$

$$V(z_1 - z_2) = |\mathbf{z}_1 - \mathbf{z}_2|\,\sigma$$

3) Area law for the average Wilson loop



$$W = P\exp i\int A_i\,dx^i \sim \exp(-\sigma\ \mathrm{Area})$$

4) Mass gap: no massless states, only massive glueballs

We shall consider quantum Yang-Mills theory at nonzero T, as we shall be interested not only in confinement at small T but also in the deconfinement phase transition at T>Tc. Quarks are switched off.

According to Feynman, the partition function is given by a path integral over all connections periodic in imaginary time, with period 1/T :

$$\mathcal{Z} \quad = \quad \int DA_\mu^a(t,\mathbf{x}) \, \exp\left(-\frac{1}{4g^2}\int_0^{\frac{1}{T}} dt \int d^3x \, F_{\mu\nu}^a F_{\mu\nu}^a\right),$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \, ,$$

$$A_\mu^a\left(t + \frac{1}{T}, \mathbf{x}\right) = A_\mu^a\,(t, \mathbf{x})\,.$$

A helpful way to estimate integrals is by the saddle point method.
Dyons are saddle points, i.e. field configurations satisfying the non-linear Maxwell equation:

$$D_\mu^{ab} F_{\mu\nu}^b = 0, \qquad D_\mu^{ab} = \delta^{ab}\partial_\mu + f^{acb} A_\mu^c \, .$$

Bogomol'nyi-Prasad-Sommerfield **monopoles** or **dyons** are self-dual configurations of the Yang-Mills field, whose asymptotic electric and magnetic fields are Coulomb-like, and the eigenvalues of the Polyakov line are *non-trivial*.

For the *SU(N)* gauge group there are *N* kinds of elementary dyons:

$$\mathbf{E} = \mathbf{B} \overset{|\mathbf{x}|\to\infty}{=} \frac{\mathbf{x}}{|\mathbf{x}|^3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

hence, "dyons"

$$L(\mathbf{x}) = \mathcal{P} \exp\left(i \int_0^{1/T} dx^4 \, A_4(\mathbf{x}, x^4) \right) \xrightarrow{|\mathbf{x}|\to\infty} \begin{pmatrix} e^{2\pi i \mu_1} & 0 & 0 \\ 0 & e^{2\pi i \mu_2} & 0 \\ 0 & 0 & e^{2\pi i \mu_3} \end{pmatrix}$$ "holonomy"

$$\mu_1 \le \mu_2 \le \mu_3 \le \mu_1 + 1, \qquad \mu_1 + \mu_2 + \mu_3 = 0.$$

Inside the dyons' cores, whose size is $\frac{1}{2\pi T(\mu_m - \mu_n)}$, the field is large and, generally, time-dependent, the non-linearity is essential. Far away the field is weak and static.

In the saddle point method, one has to compute small-oscillation determinants about classical solutions.

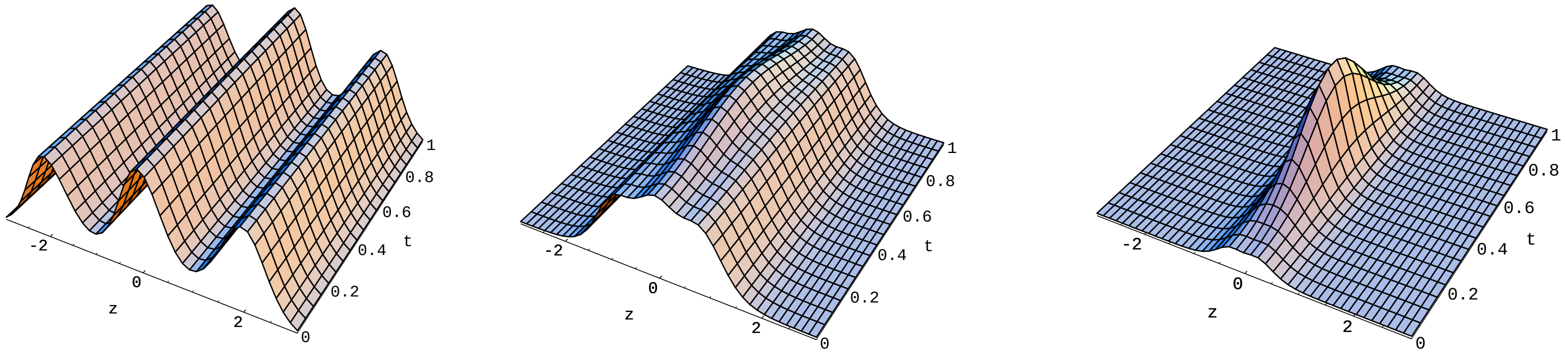
The small-oscillation determinant about a single dyon is infrared-divergent (because of the Coulomb asymptotics at infinity)

⇒ isolated dyons are unacceptable, they have zero weight

One has to take *neutral* clusters of *N* kinds of dyons. The corresponding exact solutions are known as Kraan-van Baal-Lee-Lu (KvBLL) calorons or **instantons with non-trivial holonomy** (1998).

The KvBLL instantons generalize standard instantons to the case when the Polyakov loop (the holonomy) is nontrivial, $\mu_1,\ \mu_2,\ \ldots \mu_N \neq \frac{k}{N}, \qquad k = 0, 1, ..., (N-1)$

The analytical solution shows what happens when dyons come close to each other:



Action density as function of time of three dyons of the SU(3) group.

At large dyon separations, we have three *static* dyons.

When dyons merge, they become a standard *time-dependent* instanton.

In all cases the full action is the same.

The small-oscillation determinant about KvBLL instantons is finite;
computed *exactly* by Diakonov, Gromov, Petrov, Slizovskiy (2004)
as function of

- separations between *N* dyons
- the phases of the Polyakov line $\mu_1,\ \mu_2, \ldots, \mu_N$
- temperature T
- Λ, the renormalized scale parameter

The 1-loop statistical weight (or probablity) of an instanton with non-trivial holonomy:

$$W = \int d\mathbf{x}_1 \ldots d\mathbf{x}_N \; \det G \, f^N .$$

$$f = \frac{4\pi}{g^4}\frac{\Lambda^4}{T}\, c,$$

"fugacity"

$$c = (\mathrm{Det}(-\triangle))^{-1}_{\mathrm{reg,\,norm}} \approx \exp\left(-VT^3\, P^{\mathrm{pert}}(\mu_m)\right)$$

$$G^{N\times N}_{mn} = \delta_{mn}\left(4\pi\nu_m + \frac{1}{T|\mathbf{x}_{m,m-1}|} + \frac{1}{T|\mathbf{x}_{m,m+1}|}\right) - \frac{\delta_{m,n-1}}{T|\mathbf{x}_{m,m+1}|} - \frac{\delta_{m,n+1}}{T|\mathbf{x}_{m,m-1}|}$$

$$\nu_m = \mu_{m+1} - \mu_m, \qquad \sum \nu_m = 1.$$

Gibbons and Manton (1995); Lee, Weinberg and Yi (1996); Kraan (2000); DD and Gromov

The expression for the metric of the moduli space G is exact, *valid for all separations* between dyons.

If holonomy is trivial, or T -> 0, the measure reduces to that of the standard instanton, written in terms of center, size and orientations [Diakonov and Gromov (2005)].

The perturbative potential energy (it is present even in the absence of dyons) as function of the Polyakov loop phases μ_m :

$$P^{\rm pert}(\mu_m) = \frac{(2\pi)^2\, T^3}{3} \sum_{m>n}^{N} (\mu_m - \mu_n)^2 [1 - (\mu_m - \mu_n)]^2 \Big|_{\rm mod\ 1} \quad \sim \; T^3$$

It has *N* degenerate minima when all μ_m are equal (*mod 1*) i.e. when the Polyakov loop belongs to one of the *N* elements of the group center:

$$L = e^{\frac{2\pi i k}{N}}\, {\rm diag}(1, 1 \ldots 1) \in Z_N, \qquad k = 1, 2 \ldots N\,.$$

In perturbation theory, deviation from these values are forbidden as *exp*(- const.*V*).

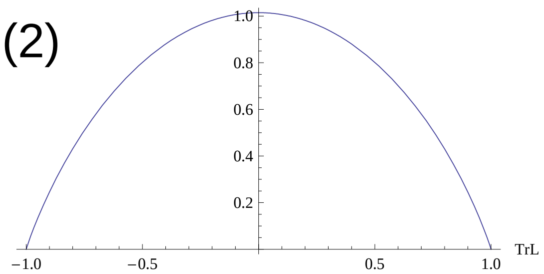
For confinement, one needs *Tr L = 0* , which is achieved at the **maximum** of the perturbative energy!

$$\mu_m^{\rm conf} = -\frac{1}{2} - \frac{1}{2N} + \frac{m}{N}$$

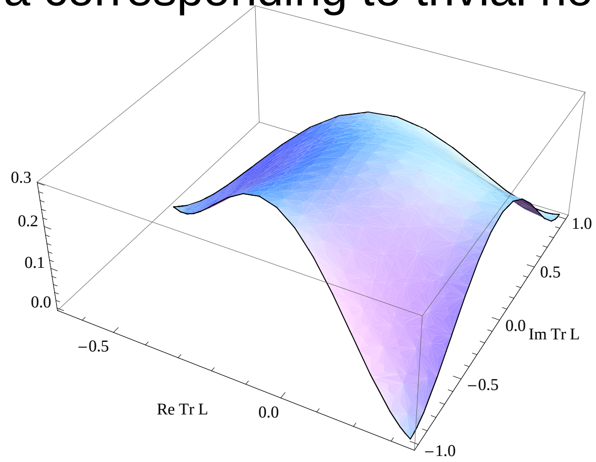
$$L = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{0\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \quad {\rm for}\ SU(3), \qquad {\rm Tr}\, L = 0.$$

Perturbative potential energy has *N* minima corresponding to trivial holonomy:

SU(2)

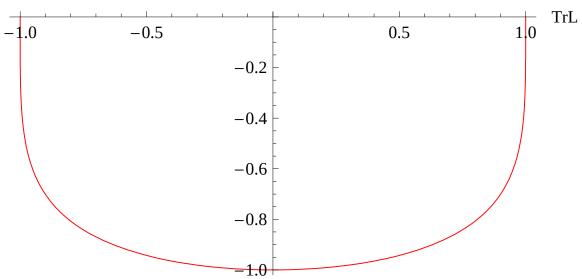


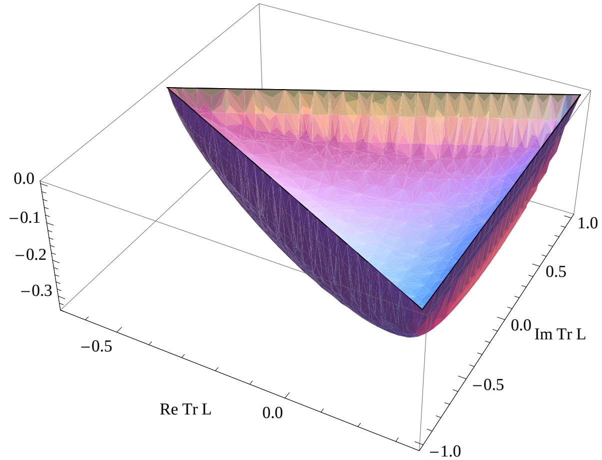
scale as T^4



SU(3)

However, the non-perturbative free energy of the ensemble of $\mathcal{O}(V)$ dyons has the **minimum** at *Tr L = 0* ! At low *T* it wins → confinement !
[DD (2003)]





At T<Tc the dyon-induced free energy prevails and forces the system to pick the “confining” holonomy

$$L = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{0\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \quad \text{for } SU(3), \qquad \mathrm{Tr}\, L = 0.$$

To see it, one has to calculate the partition function of the grand canonical ensemble of an arbitrary number of dyons of *N* kinds and arbitrary μ_m's, and then minimize the free energy in μ_m's

(and also compute the essential correlation functions).

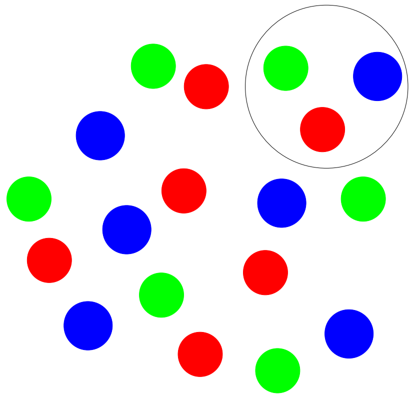
fugacity, function of T, Λ

$$\mathcal{Z} = \sum_{K_1 \ldots K_N} \frac{1}{K_1! \ldots K_N!} \prod_{m=1}^{N} \prod_{i=1}^{K_m} \int (d^3\mathbf{x}_{mi}\, f)\ \det\ G(\mathbf{x})\ .$$

moduli space metric,
function of dyon separations

K_m number of dyons of kind *m*

$\mathbf{x}_{mi}$ 3d coordinate of the *i*-th dyon of kind *m*



G is the "moduli space metric tensor" whose dimension is the total # of dyons:

$$G_{mi,nj} = \delta_{mn}\delta_{ij}\left(4\pi\nu_m + \sum_k \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{m-1,k}|} + \sum_k \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{m+1,k}|} - 2\sum_{k\neq i} \frac{1}{|\mathbf{x}_{mi} - \mathbf{x}_{mk}|}\right) - \frac{\delta_{m,n-1}}{|\mathbf{x}_{mi} - \mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_{mi} - \mathbf{x}_{m-1,j}|} + 2\,\frac{\delta_{mn}}{|\mathbf{x}_{mi} - \mathbf{x}_{mj}|}\Bigg|_{i\neq j} .$$

Properties:

1) the metric is hyper-Kaehler (a very non-trivial requirement)
2) same-kind dyons repulse each other, whereas different-kind attract e.o.
3) if dyons happen to organize into well separated neutral clusters with N dyons in each (= instantons), then *det G* is factorized
4) identical dyons are symmetric under permutations: they should not "know" what instanton they belong to!

This is an unusual statistical physics based not on the Boltzmann *exp(-U/T)* but on the measure *det G*; it can be written as *exp(Tr Log G)*, but then there will be many-body forces!

It turns out that this statistical ensemble is equivalent
to an **exactly solvable** 3d Quantum Field Theory!

Use two tricks to present the ensemble as a QFT:

1) «fermionization» [Berezin]

$$\det\, G = \int \prod_A d\psi_A^\dagger\, d\psi_A\, \exp\left(\psi_A^\dagger\, G_{AB}\, \psi_B\right)$$

anticommuting
Grassmann variables

2) «bosonization» [Polyakov]

auxiliary boson field

$$\exp\left(\sum_{m,n} \frac{Q_m Q_n}{|\mathbf{x}_m - \mathbf{x}_n|}\right) = \int D\phi\, \exp\left(-\int d\mathbf{x} (\partial_i\phi\partial_i\phi + \rho\phi)\right)$$

$$= \exp\left(\int \rho \frac{1}{\triangle} \rho\right), \qquad \rho = \sum Q_m\, \delta(\mathbf{x} - \mathbf{x}_m)$$

Here the «charges» *Q* are Grassmann variables but they can be easily
integrated out [Diakonov and Petrov (2007)]

The partition function of the dyon ensemble can be presented identically as a QFT with *2N* boson fields *v, w*, and *2N* anticommuting (ghost) fields:

$$\mathcal{Z}=\int D\chi^\dagger D\chi Dv Dw\; \exp\int d^3x\left\{\frac{T}{4\pi}\left(\partial_i\chi_m^\dagger\partial_i\chi_m+\partial_i v_m\partial_i w_m\right)\right.$$

$$\left.+\quad f\left[(-4\pi\mu_m+v_m)\frac{\partial\mathcal{F}}{\partial w_m}+\chi_m^\dagger\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\chi_n\right]\right\}$$

$$\mathcal{F}\quad=\quad\sum_{m=1}^{N}e^{w_m-w_{m+1}}\,.$$

periodic *N*-particleToda potential

$$\int Dv_m\quad\longrightarrow\quad\delta\left(-\frac{T}{4\pi}\partial^2 w_m+f\frac{\partial\mathcal{F}}{\partial w_m}\right)$$

$$\int Dw_m\,\delta\left(-\frac{T}{4\pi}\partial^2 w_m+f\frac{\partial\mathcal{F}}{\partial w_m}\right)$$

$$\longrightarrow\det{}^{-1}\left(-\frac{T}{4\pi}\partial^2\delta_{mn}+f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg|_{w=\bar{w}}\right)$$

$$\int D\chi_m^\dagger D\chi_m\quad\longrightarrow\quad\det\left(-\frac{T}{4\pi}\partial^2\delta_{mn}+f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg|_{w=\bar{w}}\right)$$

boson and ghost determinants cancel. Classical calculation is exact!

1st result , 1st criterion of confinement:

The minimum of the free energy is at equidistant values of μ_m corresponding to the zero average value of the Polyakov line!

Indeed, the dyon-induced potential energy as function of μ_m ,

$$\mathcal{P} = -4\pi f \, N \, (\nu_1 \nu_2 ... \nu_N)^{\frac{1}{N}}, \qquad \nu_1 + \nu_2 + ... + \nu_N = 1,$$

$$\nu_m = \mu_{m+1} - \mu_m$$

has the minimum at

$$\nu_1 = \nu_2 = ... = \nu_N = \frac{1}{N}, \qquad \mathcal{P}^{\min} = -4\pi f.$$

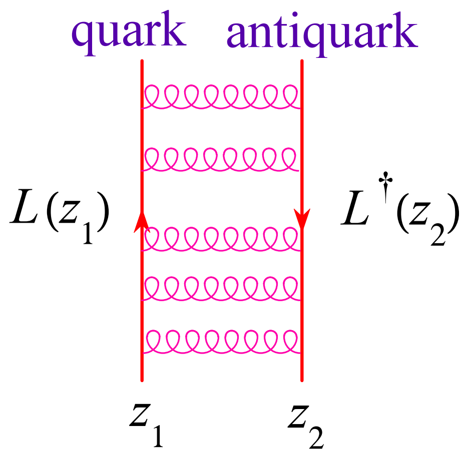
i.e. at equidistant μ_m , which implies *Tr L = 0* !

The correlation function of two Polyakov lines defines the potential energy between two static quarks:

$$\left\langle \mathrm{Tr}\, L(\mathbf{x}) \mathrm{Tr}\, L^\dagger(\mathbf{y}) \right\rangle = C \exp\left(-\frac{V(\mathbf{x}-\mathbf{y})}{T} \right)$$

$$V(\mathbf{x}-\mathbf{y}) = \sigma\, |\mathbf{x}-\mathbf{y}|, \qquad \sigma \approx 1\, \frac{\mathrm{GeV}}{\mathrm{fm}}$$

$$\sigma = \frac{\Lambda^2}{\lambda} \frac{N_c}{\pi} \sin\frac{\pi}{N_c}, \qquad N_c = 3$$



(this is the “14 tons”)

2nd result , 2^{nd} criterion of confinement:

The potential energy of static quark and antiquark is linearly rising with separation, with a calculable slope, or string tension.

The string tension has a finite limit at small T.
It is stable in the number of colours Nc, as it should be.

3d result, 3d criterion

$$W = P \exp i \oint A_i \, dx^i \sim \exp(-\sigma \, \text{Area})$$

Along the surface spanning the loop there is a large (dual) field, “the string”, leading to the area behaviour of the average Wilson loop !

At low T the “magnetic” string tension coincides with the “electric” one, as it should be: $\sigma_{\text{electr}} = \sigma_{\text{magn}}$, $T^{\text{o}} \to 0$

The Lorentz symmetry is restored, despite the 3d formulation.

Moreover, in *SU(N)* there are *N* different string tensions, classified by the “*N*-ality” of the representation, in which the Wilson loop is considered. We find

$$\sigma_{\text{electr}}(k) = \sigma_{\text{magn}}(k) = \frac{\Lambda^2}{\lambda} \frac{N_c}{\pi} \sin \frac{\pi k}{N_c}$$

the results for the two string tensions are the same although they are computed in two very different ways

for the rank-*k* antisymmetric tensor representation.

The string tension in the adjoint representation (*k=0*) is asymptotically zero.

4th result, thermodynamics of the deconfinement phase transition:

In the confinement phase, the free energy is

$$\frac{F}{V} = -N_c^2 \frac{\Lambda^4}{2\pi^2\lambda^2} + T^4 \frac{\pi^2}{45}\left(N_c^2 - \frac{1}{N_c^2}\right) - T^4 \frac{\pi^2}{45}\left(N_c^2 - 1\right)$$

dyon-induced | perturbative energy at maximum | Stefan-Boltzmann

$\mathcal{O}(N_c^2)$ gluons are cancelled from the free energy, as it should be in the confining phase!

The 1st order confinement-deconfinement phase transition is expected at

$$T_c^4 = \frac{45}{2\pi^4} \frac{N_c^4}{N_c^4 - 1} \frac{\Lambda^4}{\lambda^2}$$

(At Nc = 2 the free energy depends only on one variable, and the phase transition is explicitly 2nd order, in agreement with the lattice data.)

Critical temperature T_c in units of the string tension for various numbers N_c :

	$N_c = 3$	4	6	8
$T_c/\sqrt{\sigma}$, theory	0.6430	0.6150	0.5967	0.5906
$T_c/\sqrt{\sigma}$, lattice	0.6462(30)	0.6344(81)	0.6101(51)	0.5928(107)

[lattice data: Lucini, Teper and Wenger (2003)]

Another important quantity characterizing the non-perturbative vacuum –
the “topological susceptibility” :

$$\frac{(< Q_T^2 >)^{\frac{1}{4}}}{\sqrt{\sigma}} = \begin{cases} 0.439, & \text{theory} \\ 0.434(10), & \text{lattice} \end{cases} \qquad \text{for } N_c = 3.$$

Why does the semiclassical picture work so suspiciously well?

Landau (allegedly): “If there is no small parameter the problem is not formulated”

Contribution from a classical saddle point (here: dyon), schematically

$$\underbrace{M^4 \exp\left(-\frac{2\pi}{\alpha_s}\right)}_{\Lambda^4} \left[1 + c_1 \underbrace{\frac{\alpha_s N}{2\pi}}_{\lambda} + c_2 \left(\frac{\alpha_s N}{2\pi}\right)^2 + \dots\right]$$

α_s (or λ) is the running coupling constant whose argument is the maximal

scale in the problem, i.e. $\max\left(\mathrm{density}^{\frac{1}{3}}, \mathrm{temperature}\right)$. At $T \approx T_c \approx \Lambda$

$$\frac{1}{\lambda} \equiv \frac{2\pi}{\alpha_s N} = \frac{11}{3} \ln\left(\frac{4\pi T}{\Lambda e^{\gamma_E}}\right)\Bigg|_{T\approx T_c\approx\Lambda} \approx 7.$$

Therefore, the loop expansion goes in powers of ~ 1/7 .

If the qualitative picture is captured correctly, one can systematically improve the accuracy later.

Summary

1) The statistical weight of gluon field configurations in the form of *N* kinds of dyons has been computed exactly to 1-loop

2) Statistical physics of the ensemble of interacting dyons is governed by an exactly solvable 3d QFT

3) The ensemble of dyons self-organizes in such a way that all criteria of confinement are fulfilled

Non-trivial holonomy allows the existence of dyons,
dyons request the holonomy to be maximally non-trivial !

4) All quantities computed are in good agreement with lattice data

5) A simple picture of a semi-classical vacuum based on dyons works surprisingly well!