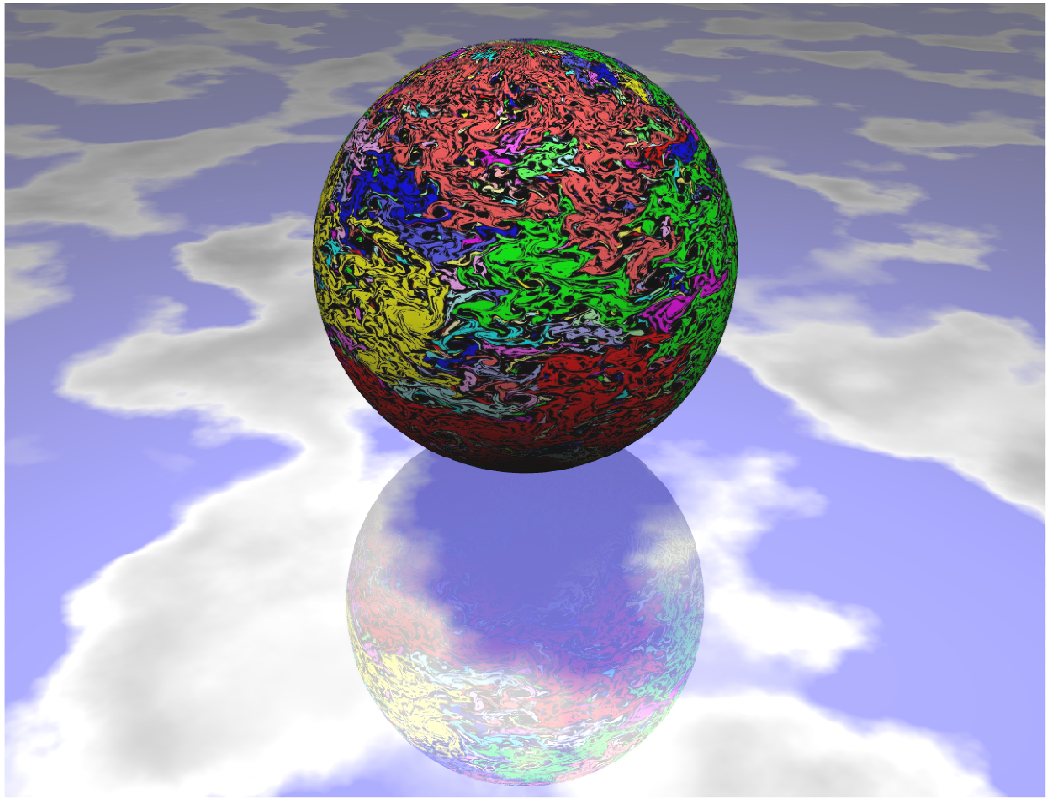
Emerging symmetries and condensates in turbulent inverse cascades

Gregory Falkovich
Weizmann Institute of Science



Landau 100, Chernogolovka, June 23

probability distribution $\mathcal{P}(\delta v, \mathbf{r})$

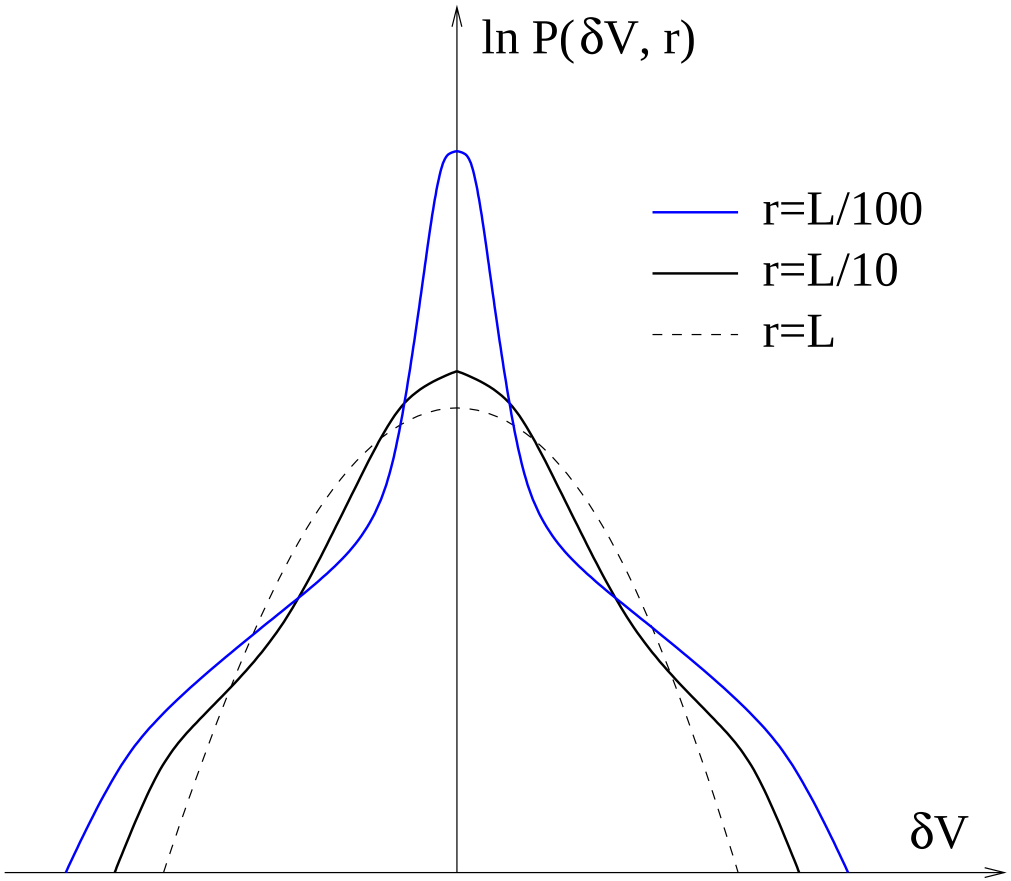
of the longitudinal velocity difference $\delta v = (\delta \mathbf{v} \cdot \mathbf{r})/r$

One asks whether the distribution is

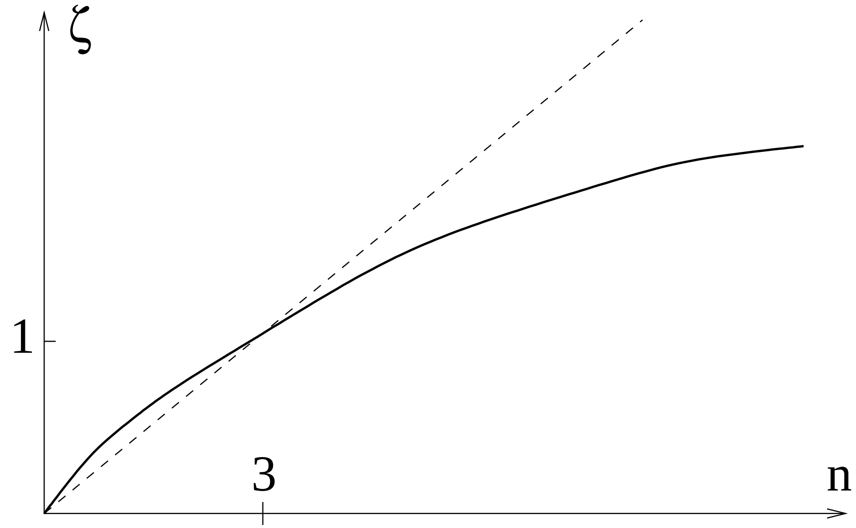
- time reversible, $\mathcal{P}(\delta v, \mathbf{r}) = \mathcal{P}(-\delta v, \mathbf{r})$,
- isotropic, $\mathcal{P}(\delta v, \mathbf{r}) = \mathcal{P}(\delta v, r)$,
- scale invariant, $\mathcal{P}(\delta v, r) = (\delta v)^{-1} f(\delta v / r^h)$

Lack of scale-invariance in direct turbulent cascades

$$P(\delta v, r) \stackrel{?}{=} (\delta v)^{-1} g[\delta v/(\epsilon r)^{1/3}]$$



$$S_n = \langle \delta v^n \rangle = C_n r^{\zeta_n}$$

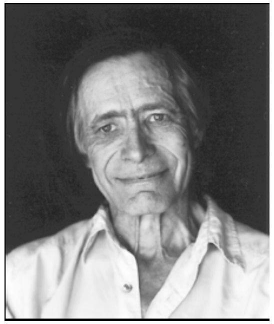


2d Navier-Stokes equations

$$\omega = \nabla \times \mathbf{u}$$

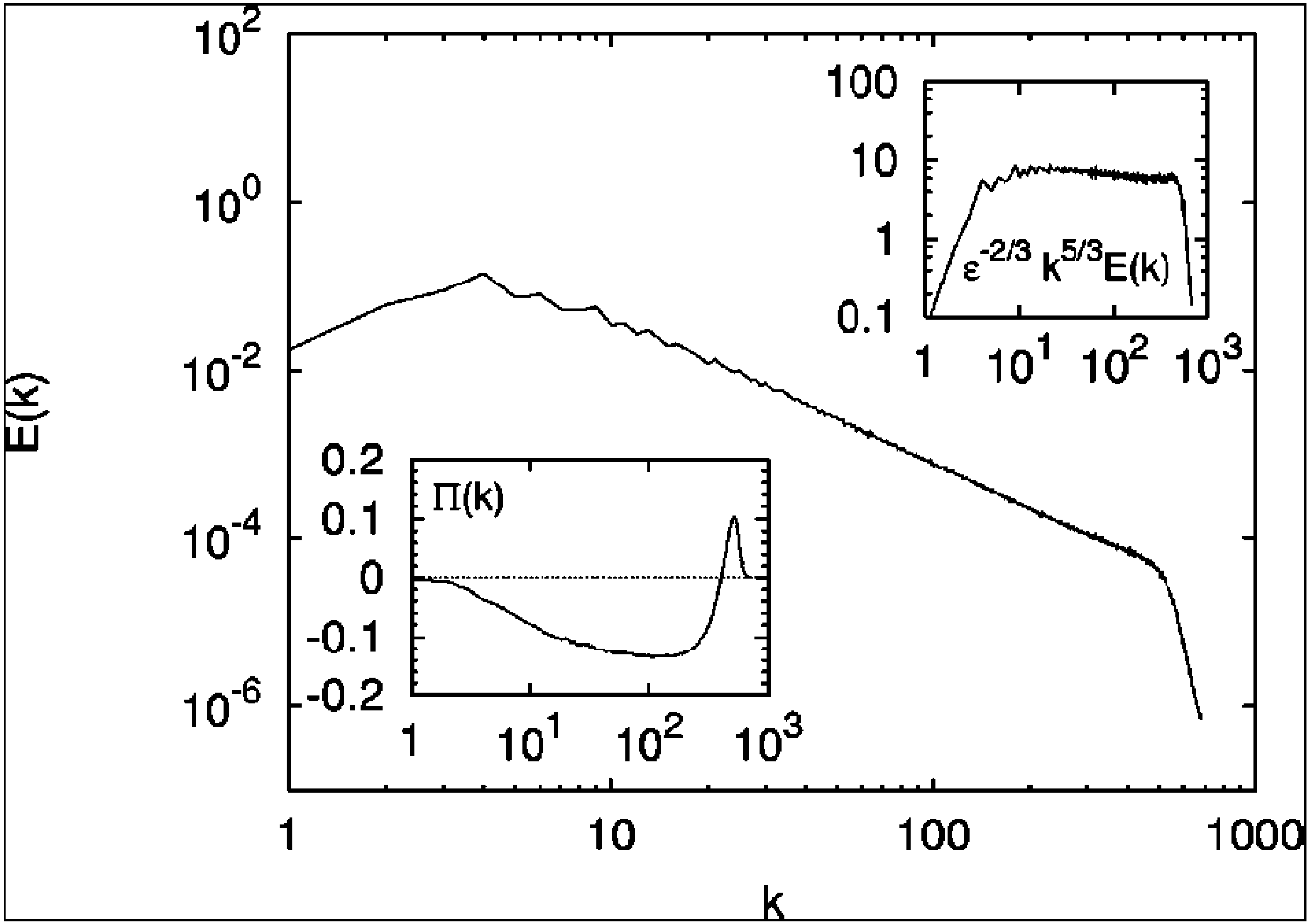
$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega - \alpha \omega + \nabla \times f$$

Kraichnan 1967



$$E = \frac{1}{2}\int |\mathbf{u}|^2 d^2x$$

$$Z = \frac{1}{2}\int \omega^2 d^2x$$



Family of transport-type equations

$$\partial a/\partial t + (\mathbf{v}\cdot\nabla)a = f + \nu\Delta a - \alpha a \qquad (*)$$

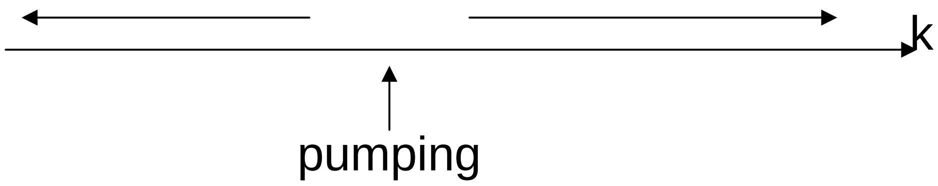
$$\mathbf{v} = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$$

$$\Psi(\boldsymbol{r},t) = \int d\boldsymbol{r}'\,|\boldsymbol{r}-\boldsymbol{r}'|^{m-2}a(\boldsymbol{r}',t)$$

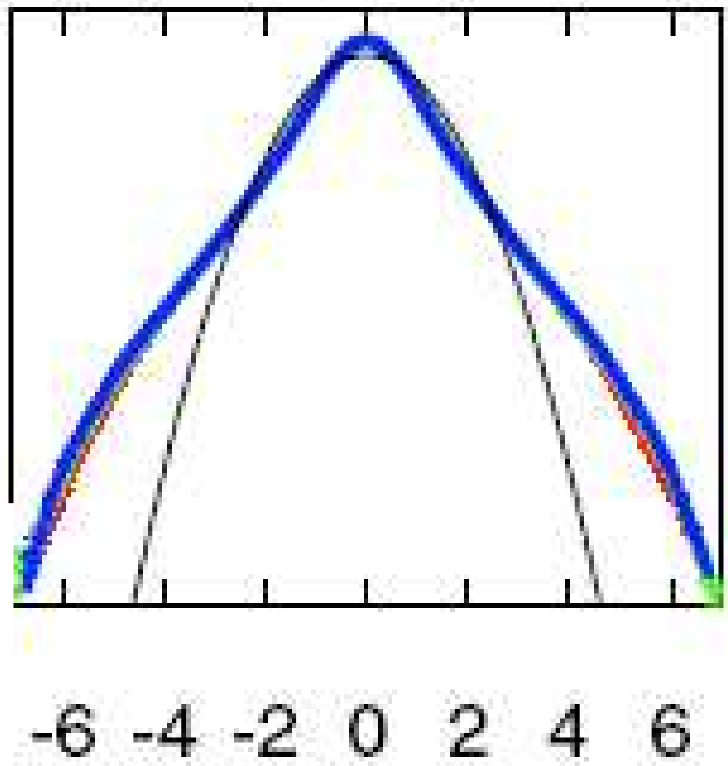
m=2 Navier-Stokes
m=1 Surface quasi-geostrophic model,
m=-2 Charney-Hasegawa-Mima model

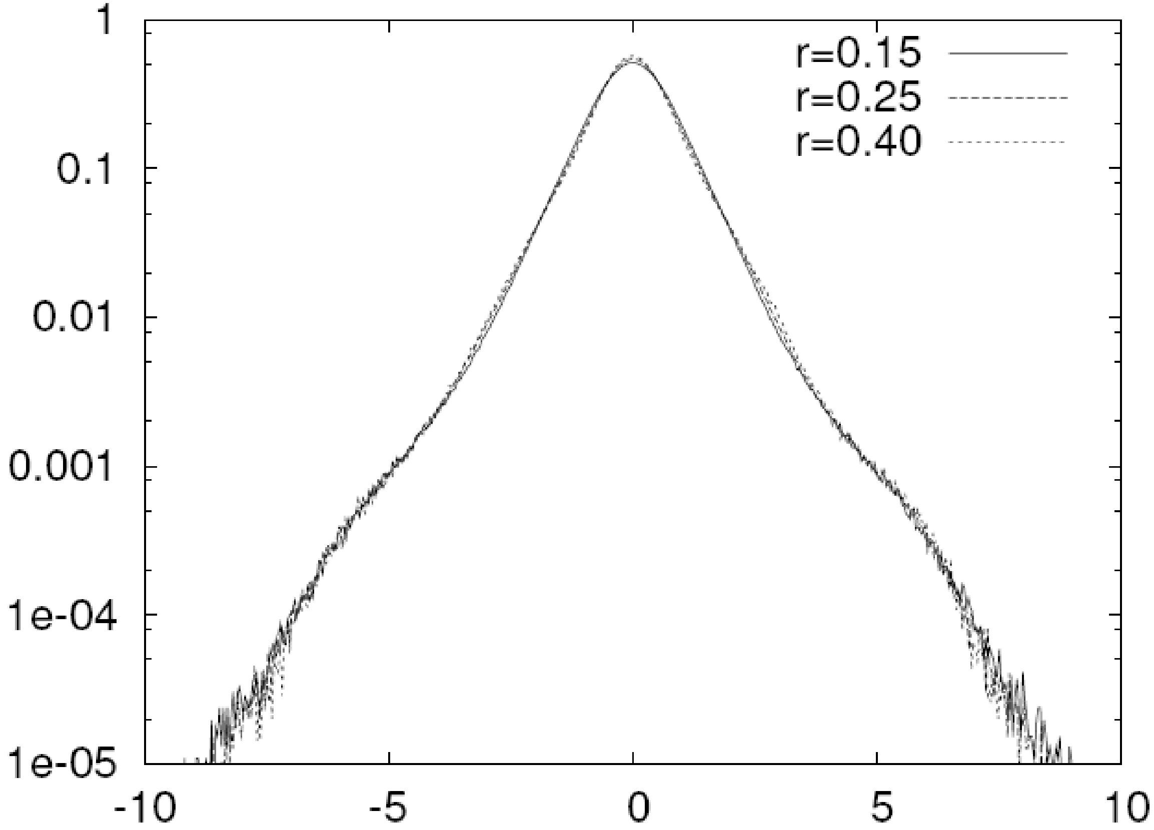
Electrostatic analogy: Coulomb law in d=4-m dimensions

lhs of (*) conserves $\sum |a_{\mathbf{k}}|^2 k^{-m}$ $\quad\sum |a_{\mathbf{k}}|^2$



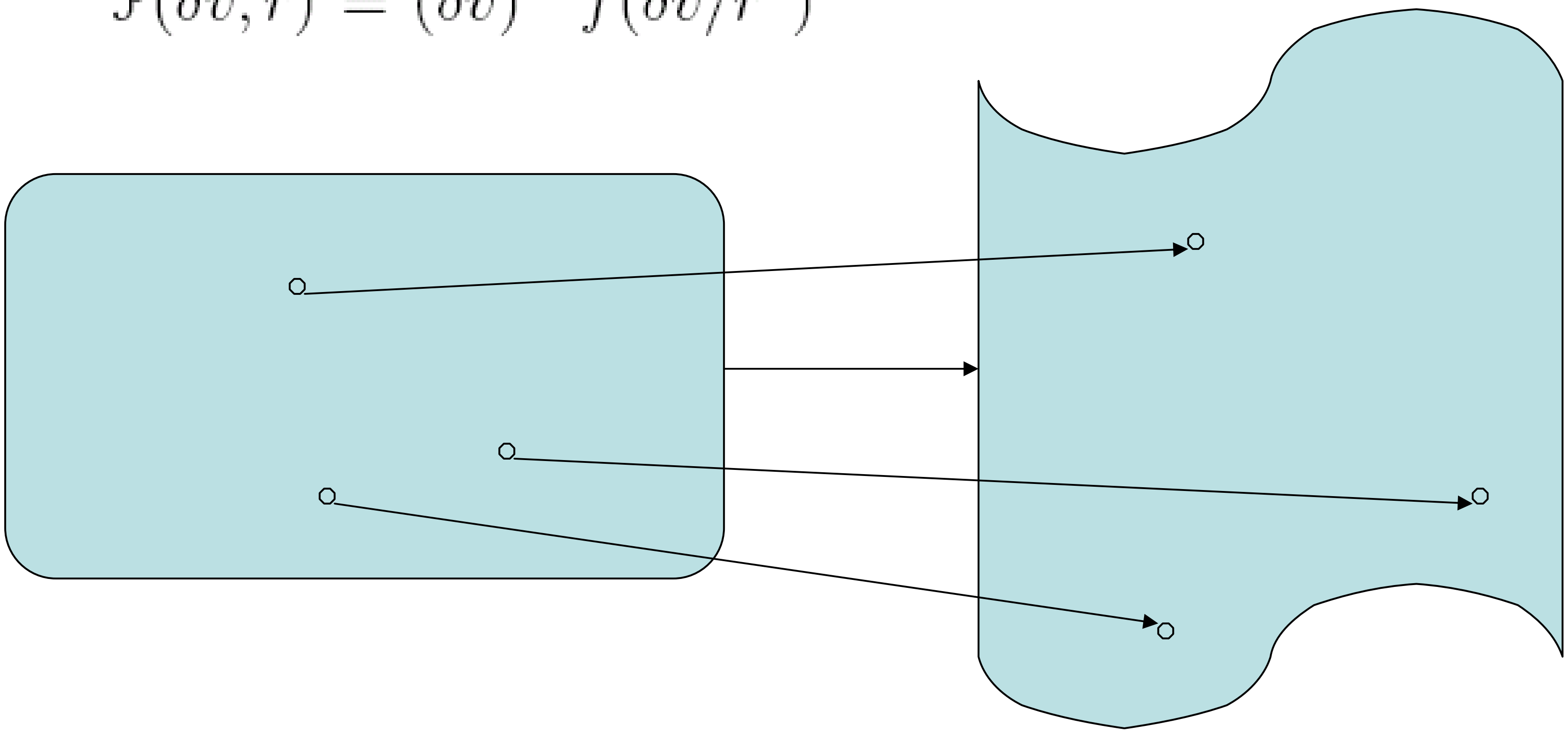
Small-scale forcing – inverse cascades





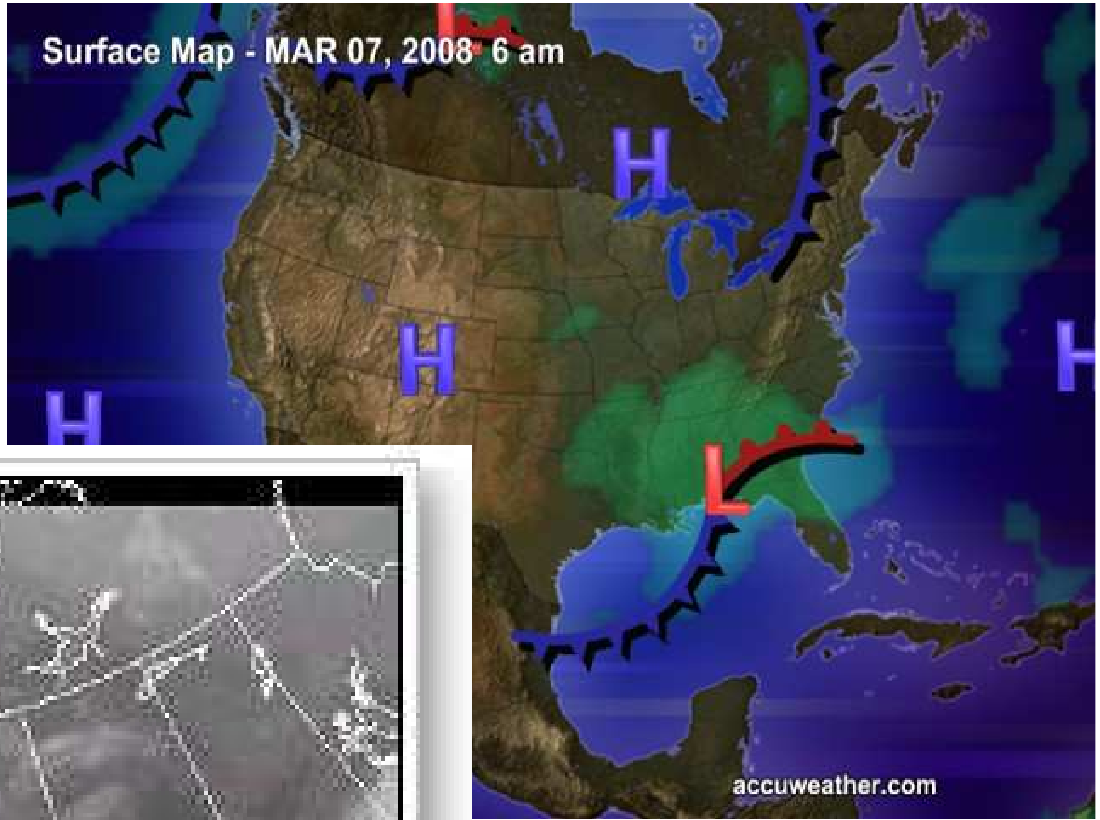
Strong fluctuations –
many interacting degrees of freedom → scale invariance.
Locality + scale invariance → conformal invariance

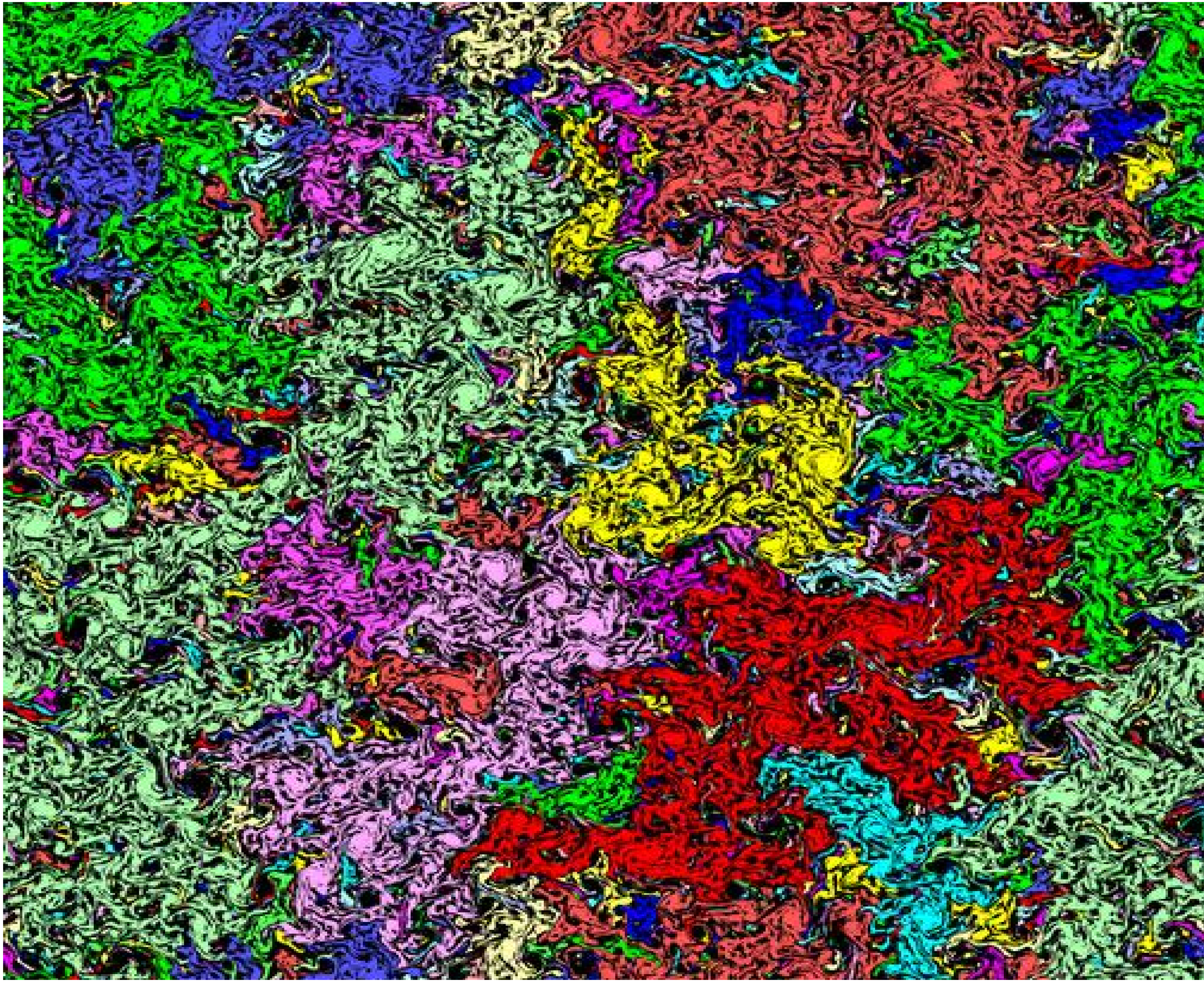
$$\mathcal{P}(\delta v, r) = (\delta v)^{-1} f(\delta v / r^h)$$



$$\mu_{\mathcal{D}}(z_1, \ldots, z_n) = \mu_{\mathcal{D}'}(f(z_1), \ldots, f(z_n))$$

Polyakov 1993


Surface Map - MAR 07, 2008 6 am
accuweather.com



Kolmogorov-Kraichnan scaling in 2d.

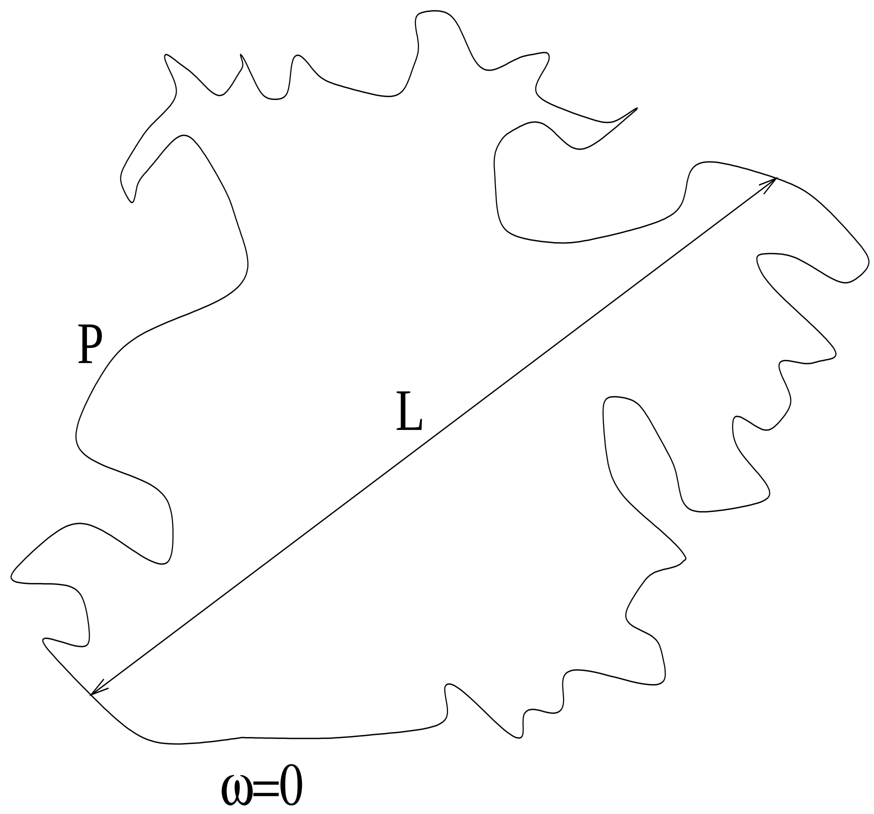
$$\frac{\text{kinetic energy } v_r^2}{\text{time } r/v_r} = \text{energy flux, } \epsilon$$

$$v_r^3 \sim \epsilon r$$

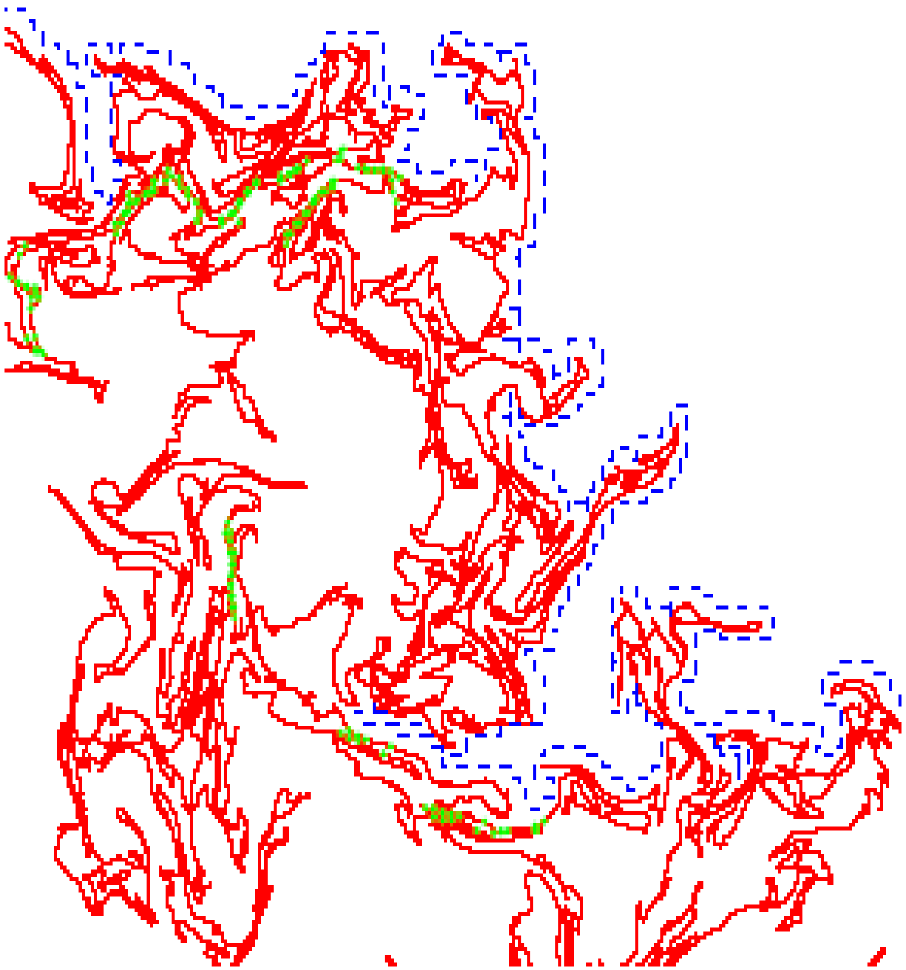
$$\int \omega dS \sim \omega_L L^2 \quad \propto L^{4/3}$$

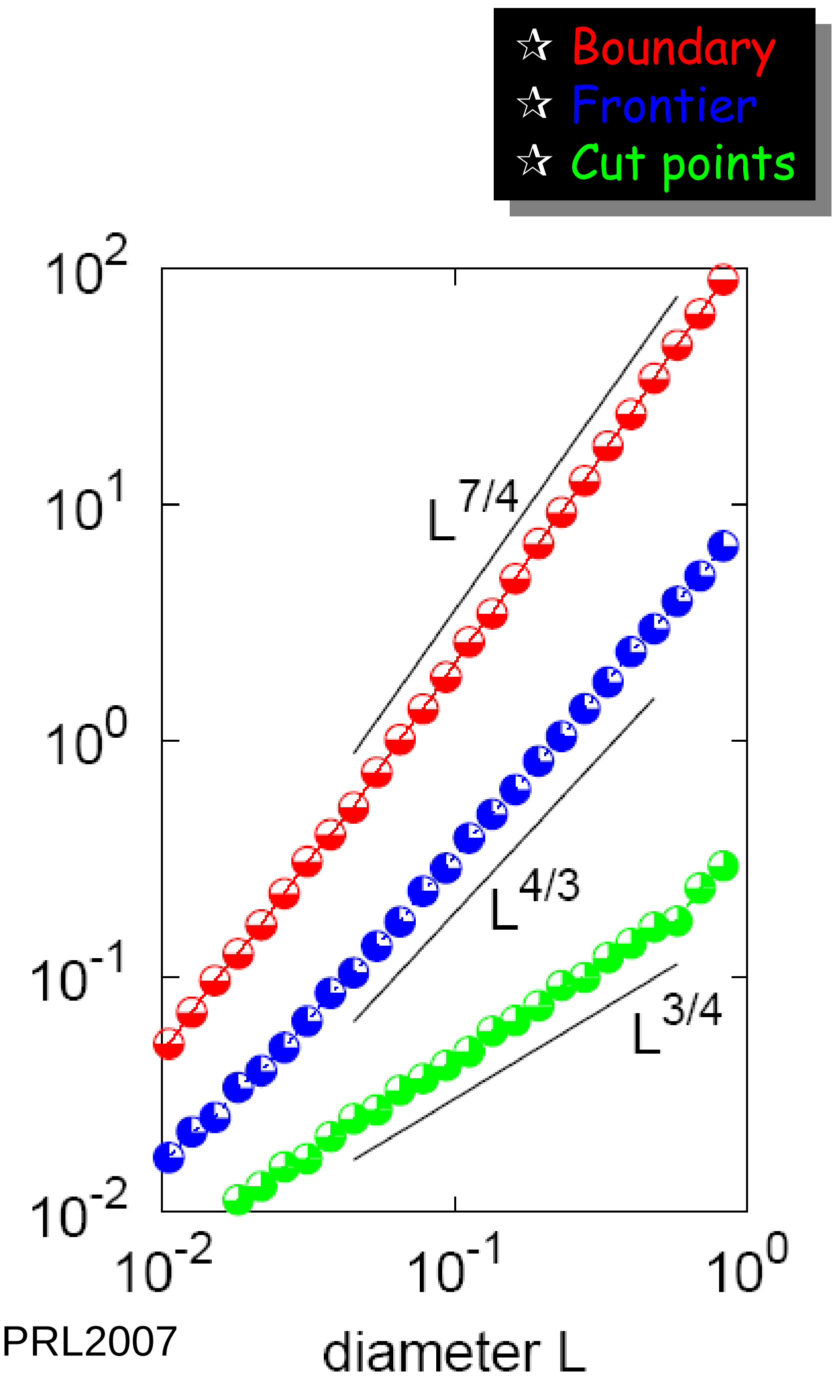
$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{\ell} \ \propto P$$

$$P \propto L^{4/3}$$



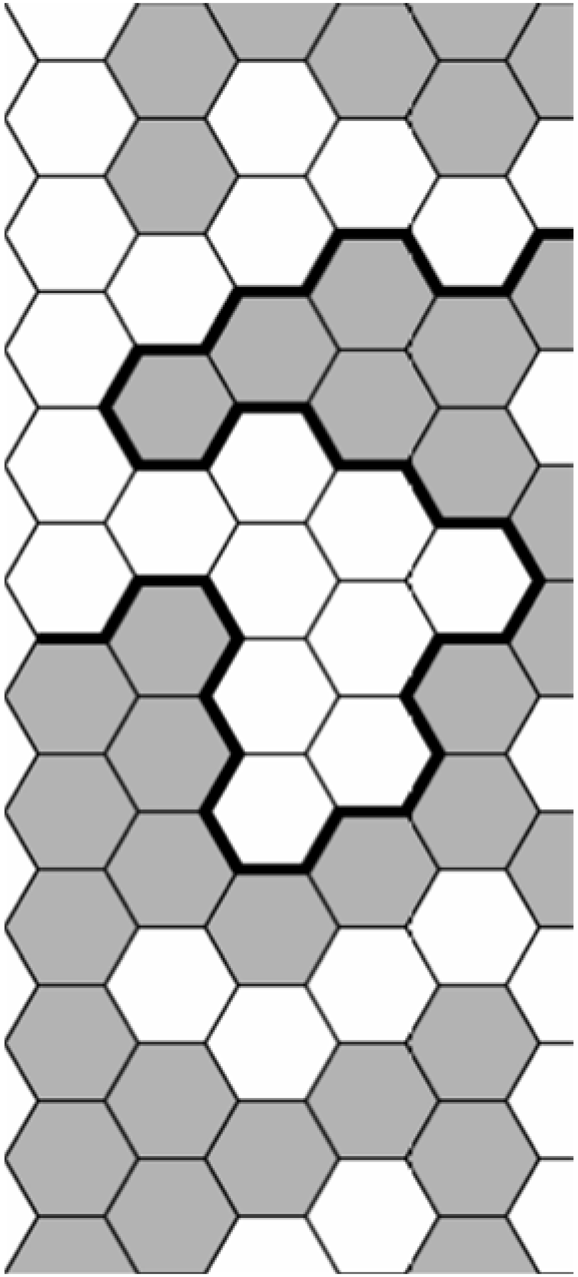
P



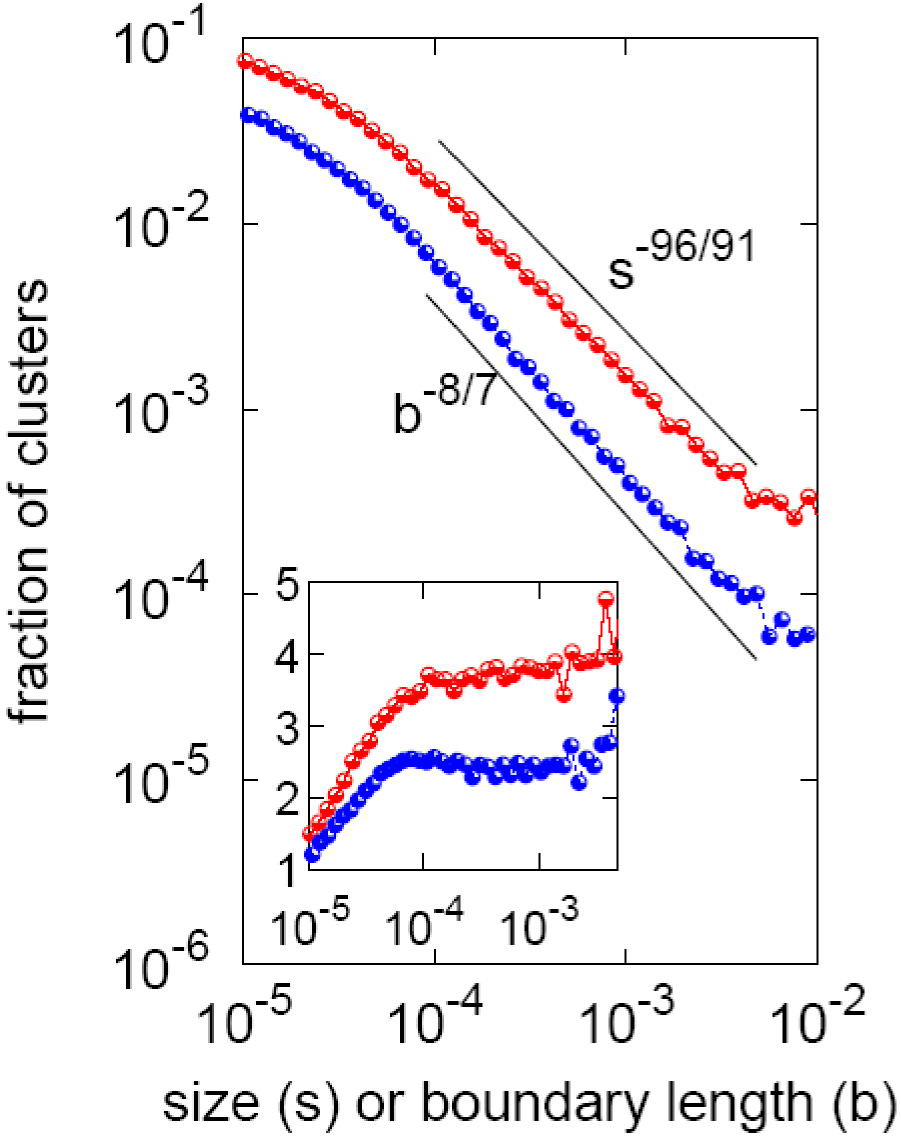


Bernard, Boffetta, Celani &GF, Nature Physics 2006, PRL2007

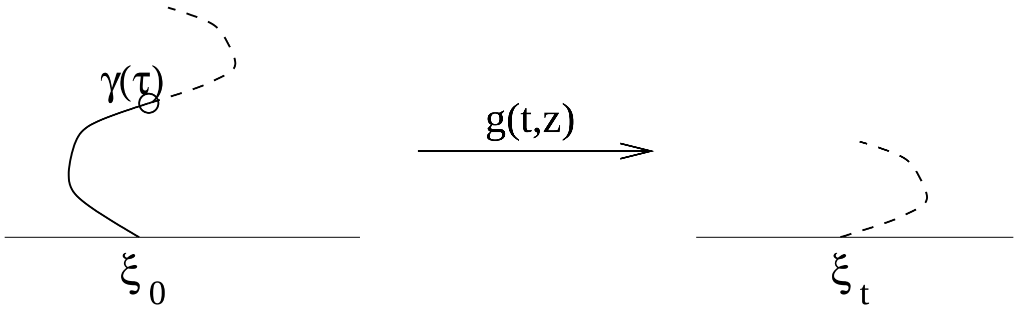
Critical Percolation



Vorticity clusters



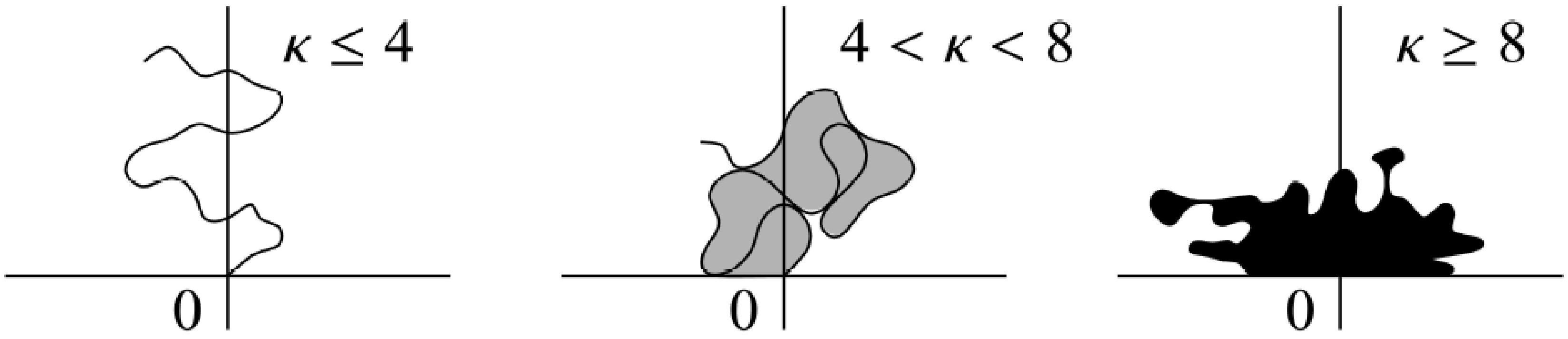
Schramm-Loewner Evolution (SLE)



$$g_t(z) \sim z + 2t/z + O(1/z^2) \text{ at infinity.}$$

$$dg_t(z)/dt = 2[g_t(z) - \xi(t)]^{-1}$$

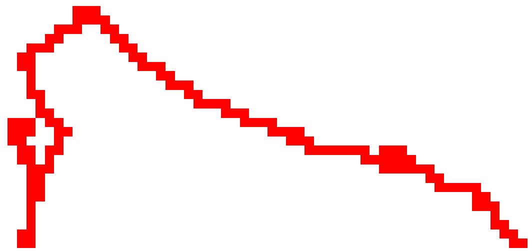
$$\langle(\xi(t) - \xi(0))^2\rangle = \kappa t.$$

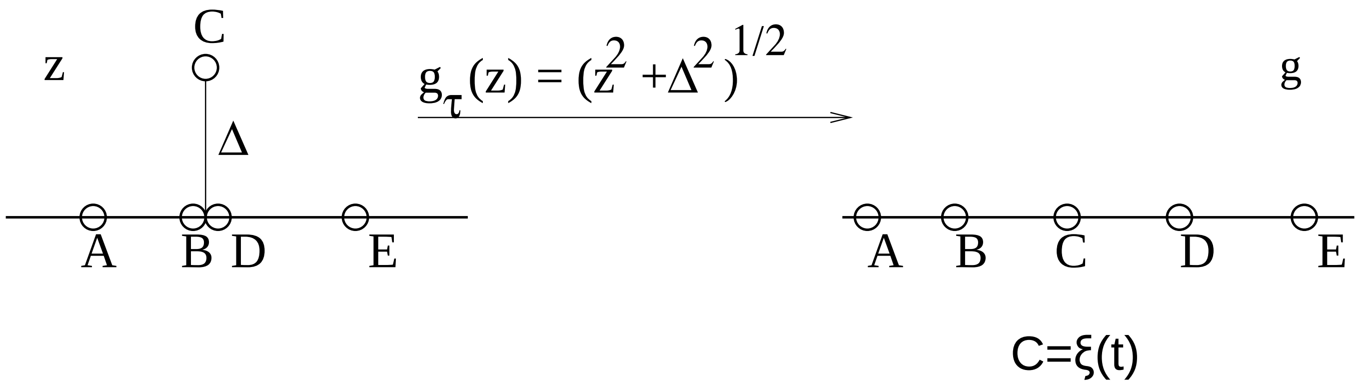


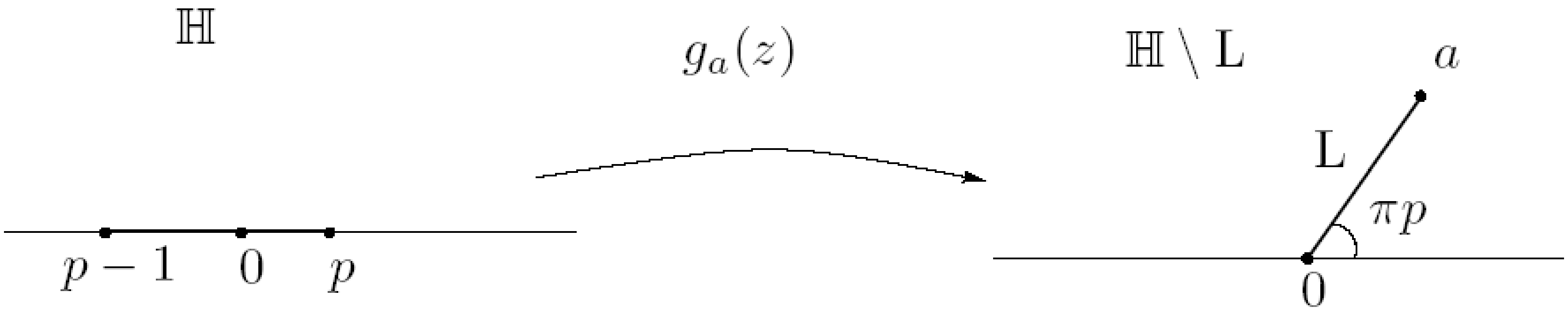
approximate $g_t(z)$ by a composition of

discrete, conformal slit maps

that swallow one segment of the curve at a time







$$g_a(z) = C(z-p)^p(z+1-p)^{1-p}$$

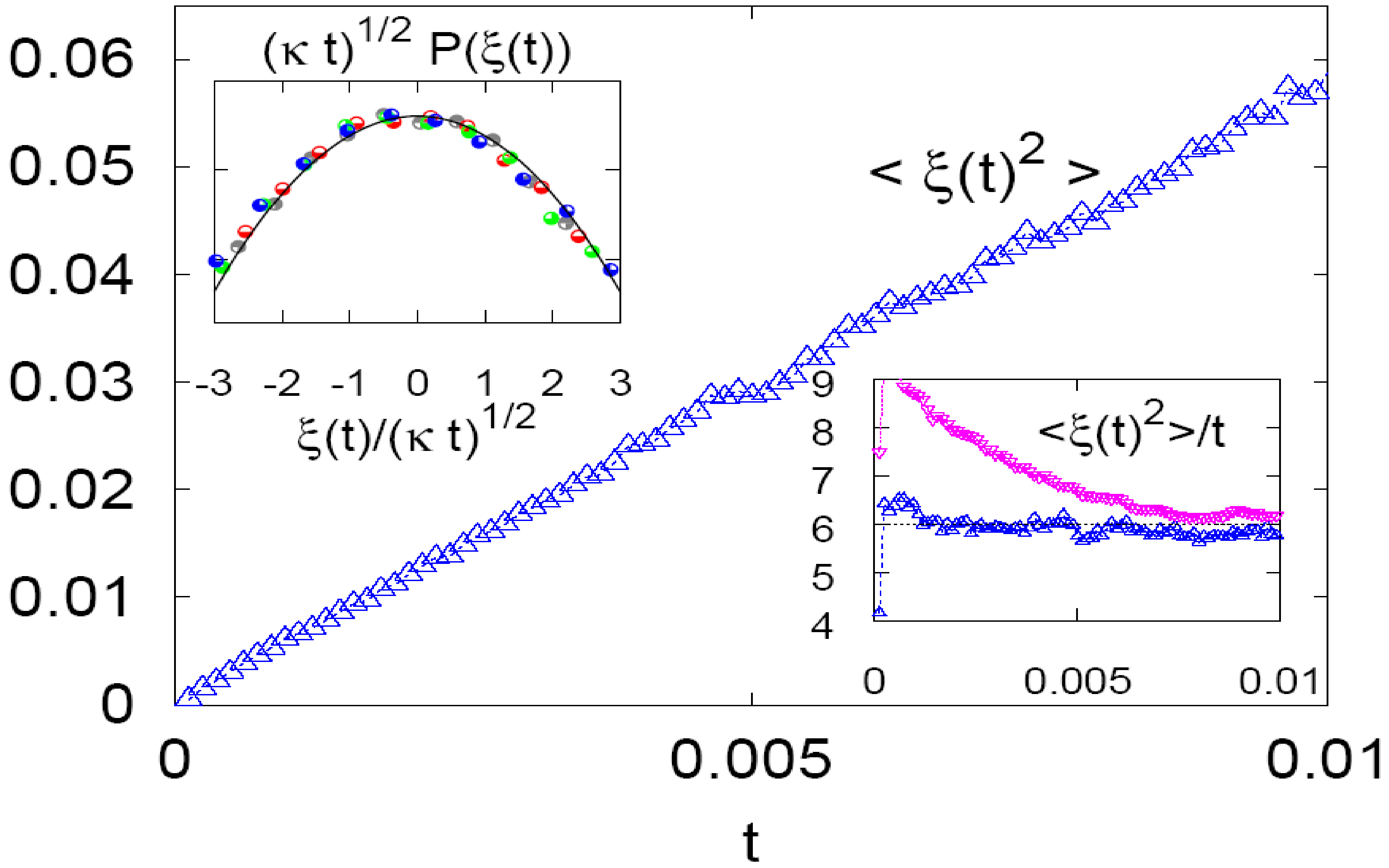
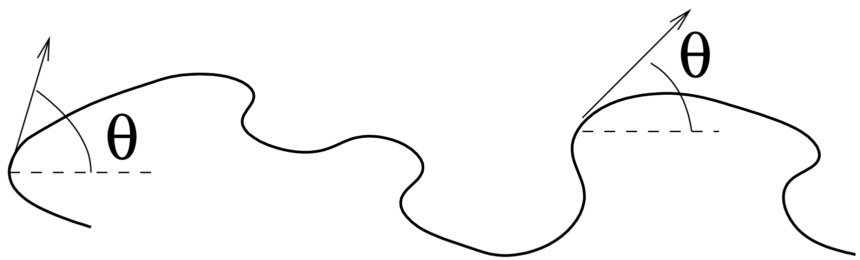
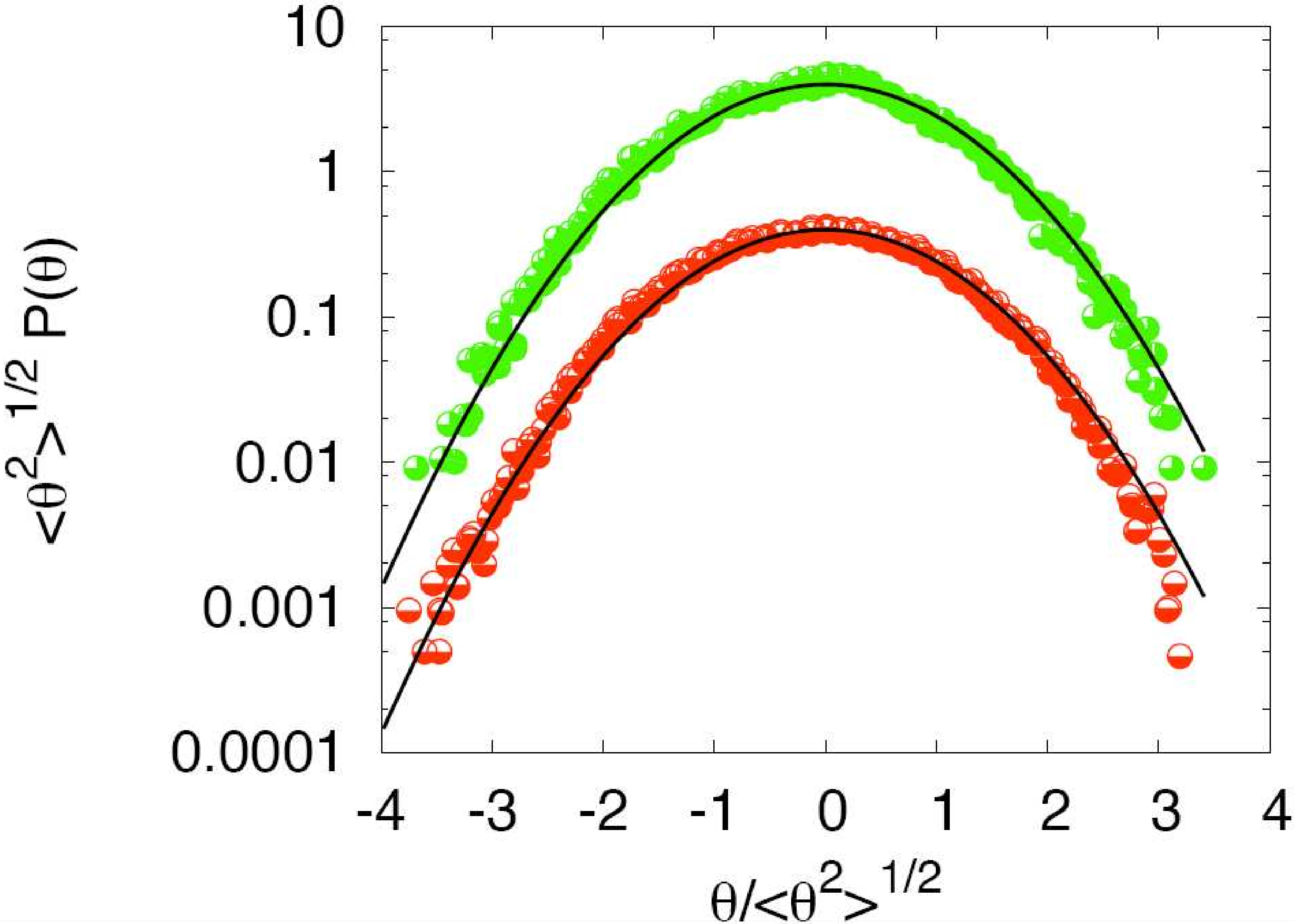
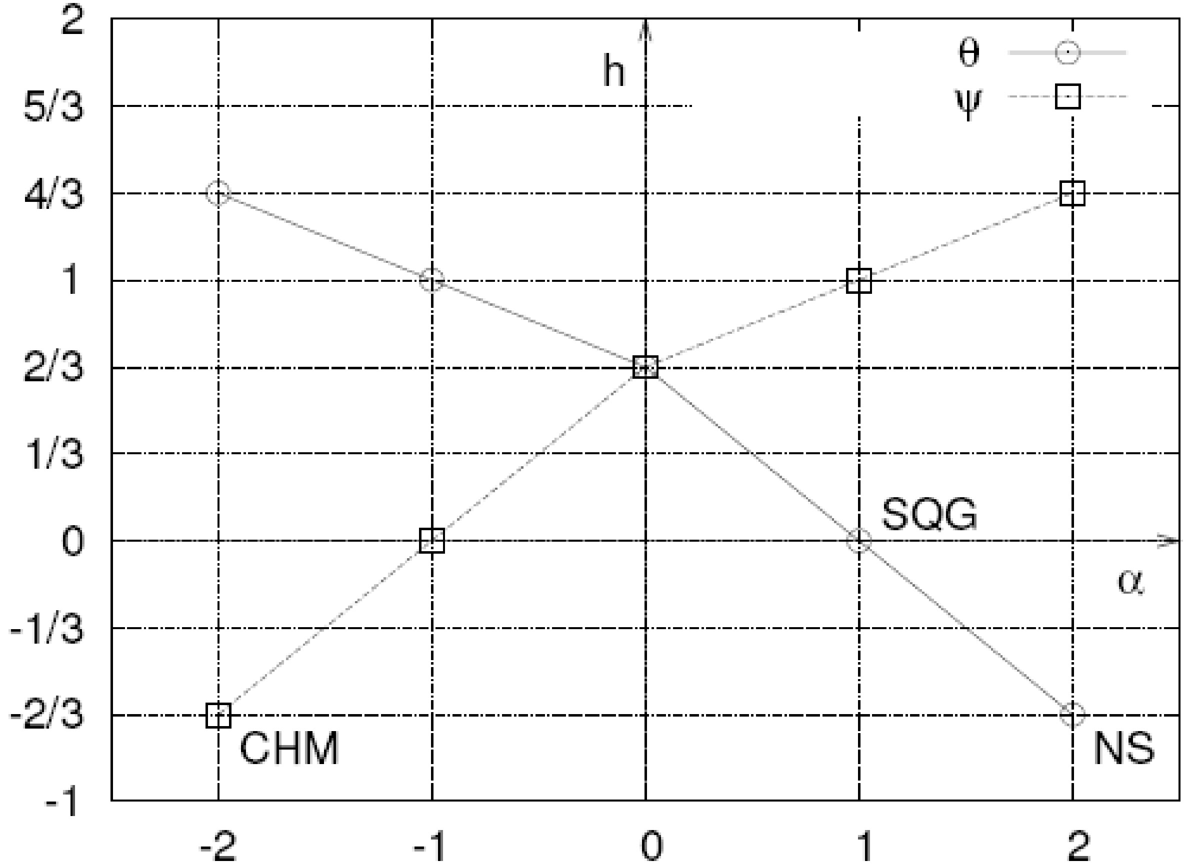


FIG. 4: The driving function is an effective diffusion process with diffusion coefficent $\kappa = 6 \pm 0.3$. The inverse cascade range corresponds to $5 \cdot 10^{-5} < t < 10^{-2}$. *Main frame*: the linear behaviour of $\langle \xi(t)^2 \rangle$. *Lower-right inset*: Diffusivity: blue for vorticity isolines, pink for the field with randomized phases. *Upper-left inset*: the probability density function of the rescaled driving function $\xi(t)/\sqrt{\kappa t}$ at four different times $t = 0.0012, 0.003, 0.006, 0.009$; the solid line is the Gaussian distribution $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.


θ
θ


10
1
0.1
0.01
0.001
0.0001
-4
-3
-2
-1
0
1
2
3
4
$<\theta^2>^{1/2} P(\theta)$
$\theta/<\theta^2>^{1/2}$


2
5/3
4/3
1
2/3
1/3
0
-1/3
-2/3
-1
-2
-1
0
1
2
h
α
θ
ψ
SQG
CHM
NS

Different systems producing SLE

- Critical phenomena with local Hamiltonians
- Random walks, non necessarily local
- Inverse cascades in turbulence
- Nodal lines of wave functions in chaotic systems
- Spin glasses
- Rocky coastlines

Bose-Einstein condensation and optical turbulence
Gross-Pitaevsky equation

$$i\psi_t + \Delta\psi + \lambda|\psi|^2\psi = 0$$

$$\langle \psi_k \psi^*_{k'} \rangle = n_k \Delta_{\mathbf{kk'}} \qquad n_k = f(k)\,T/k^2$$

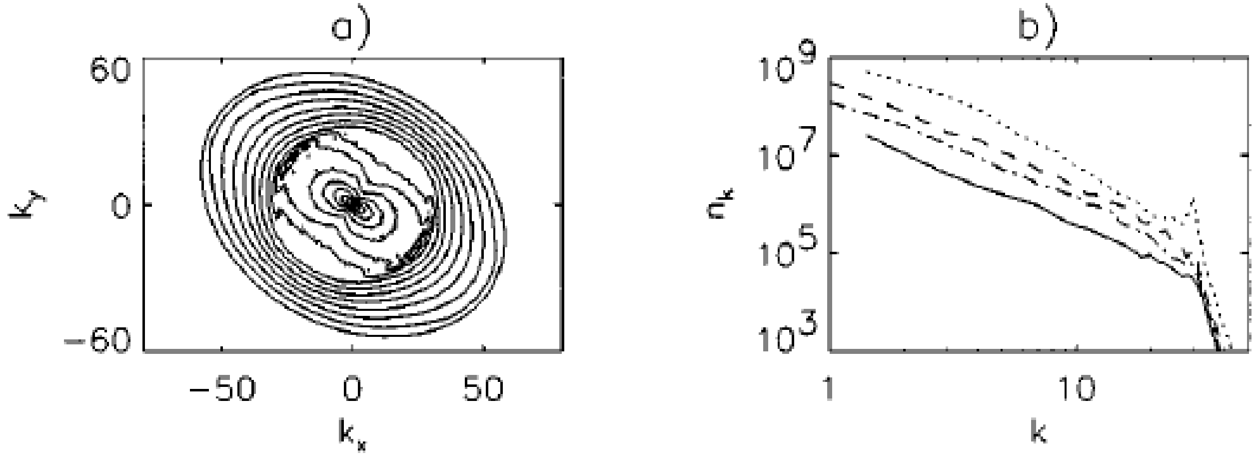
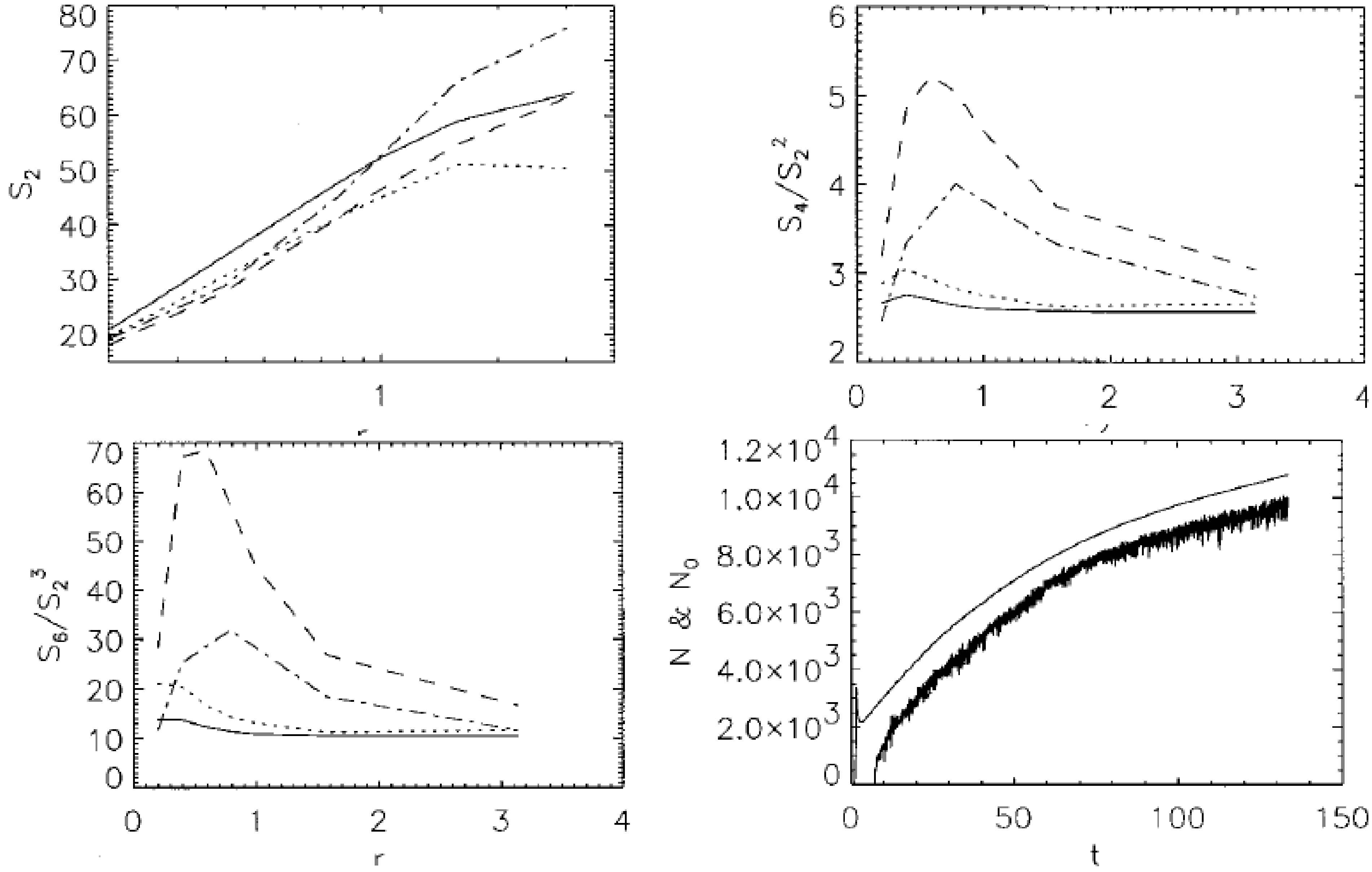
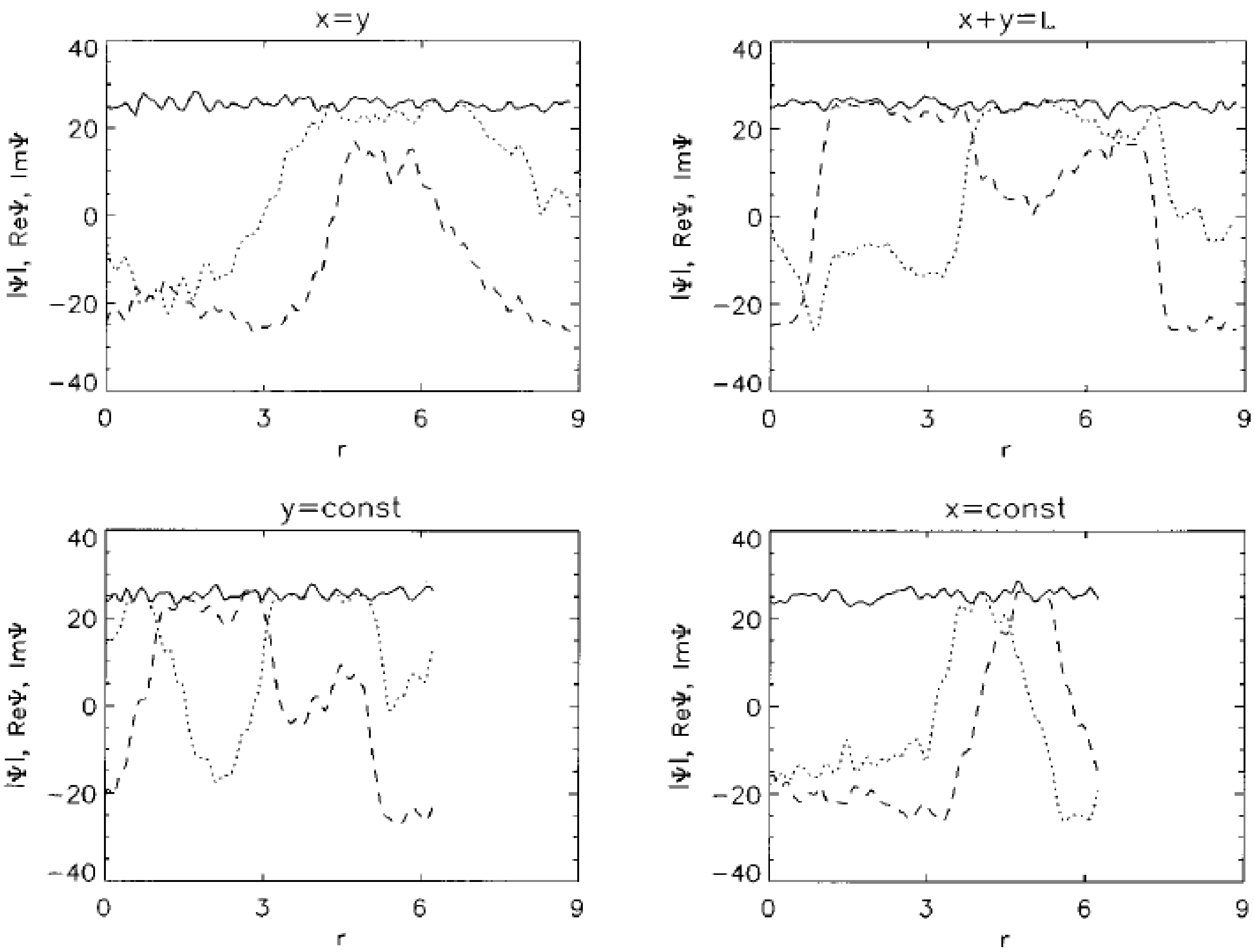


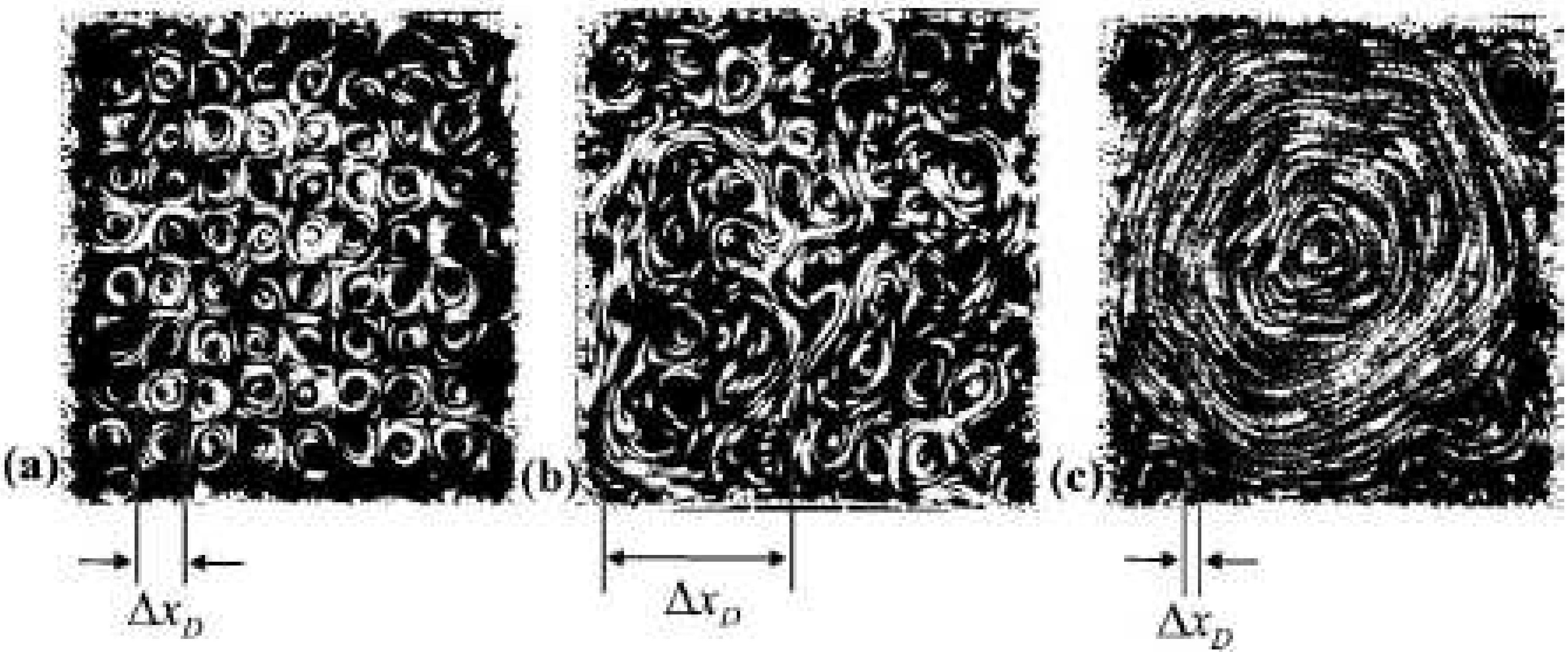
FIG. 2. (a) Level sets of the spectrum $n(k_x, k_y)$ averaged over time $t=(119\text{–}127)$. (b) Spectra at different directions in k space: $-\cdot-$ $k_x=0$, $---$ $k_y=0$, $\cdots$ $k_x=-k_y$, unbroken line $k_x=k_y$.

$$S_i(r) = \langle |\psi(\mathbf{x}+\mathbf{r}) - \psi(\mathbf{x})|^i \rangle$$



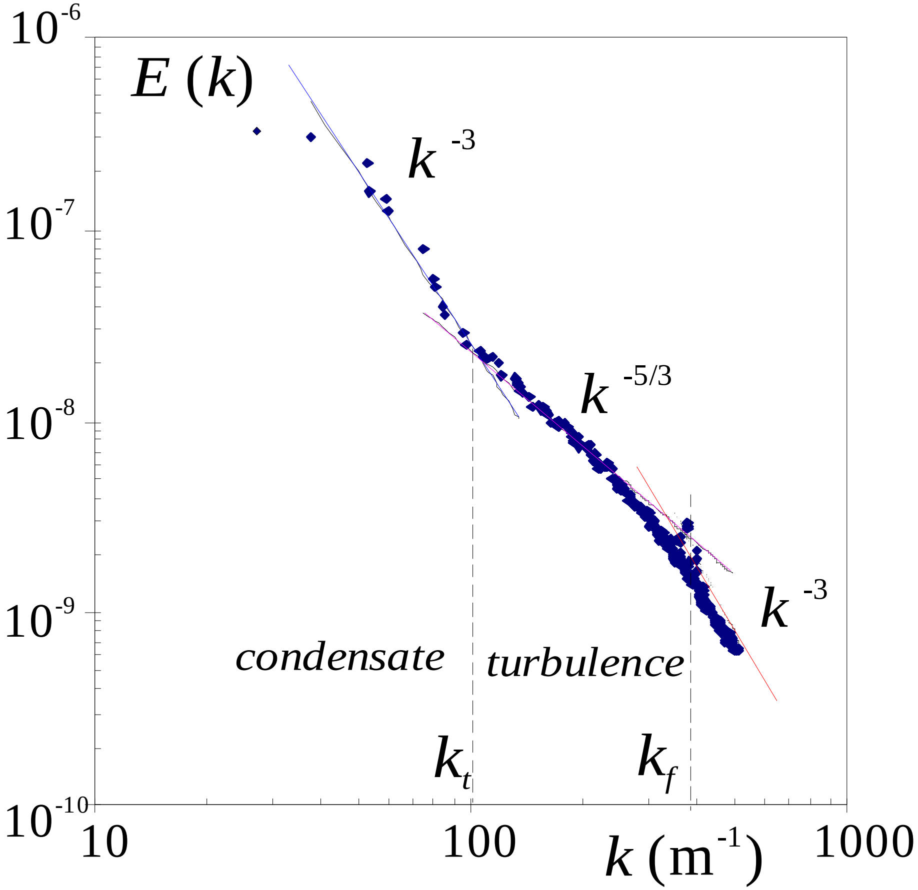


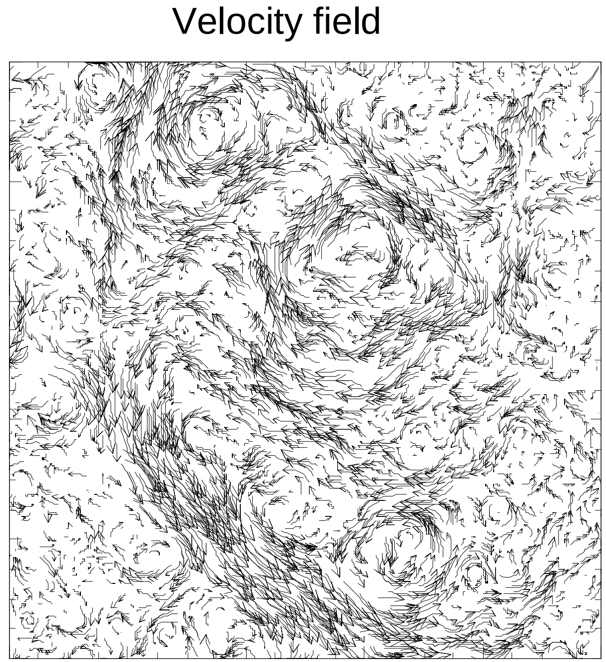
x=y
x+y=L
y=const
x=const
|Ψ|, ReΨ, ImΨ
r
40
20
0
−20
−40
0
3
6
9



(a)
Δx_D
(b)
Δx_D
(c)
Δx_D

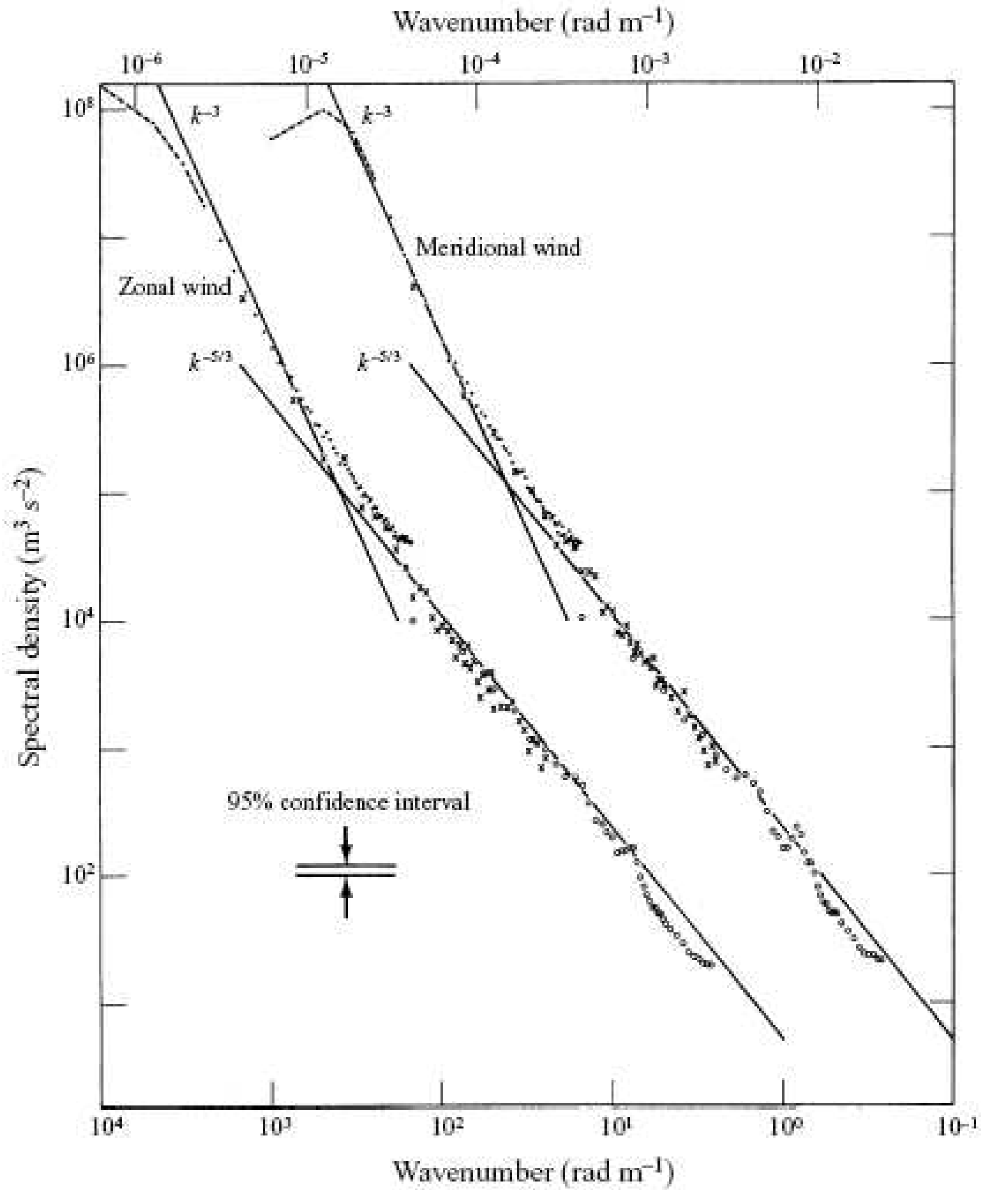
inverse cascades lead to emerging symmetries but eventually to condensates which break symmetries in a different way for different moments





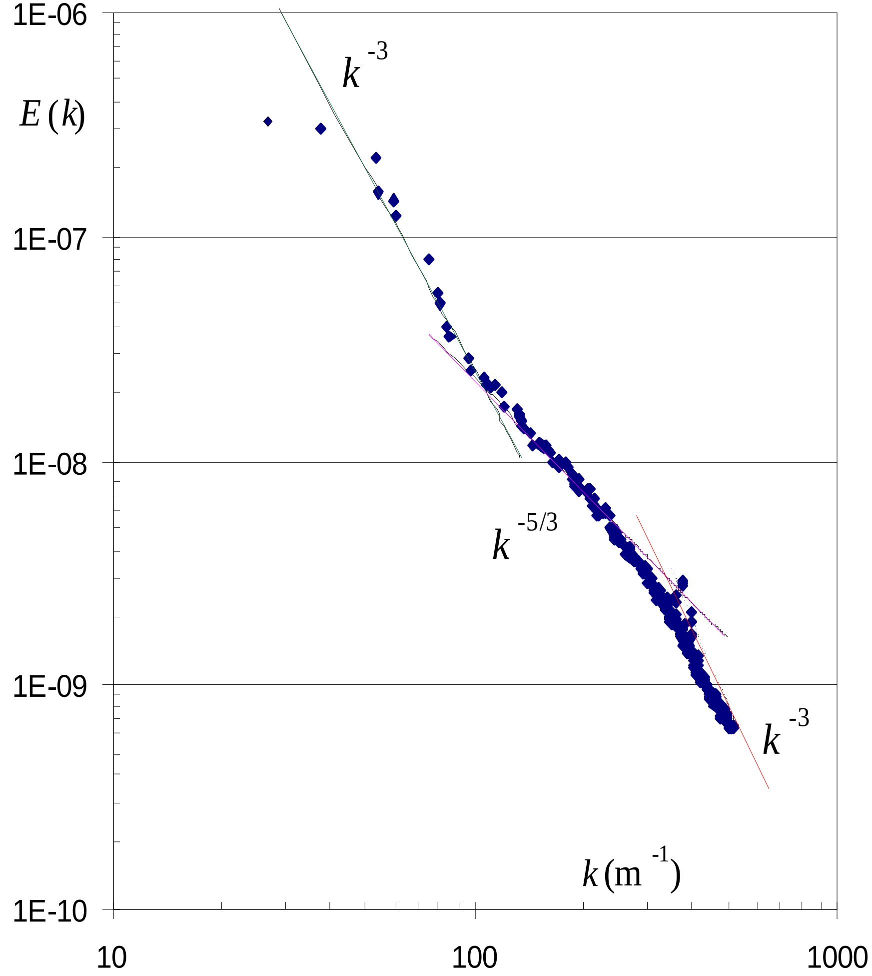
Atmospheric spectrum

Nastrom, Gage, J. Atmosph. Sci. 1985

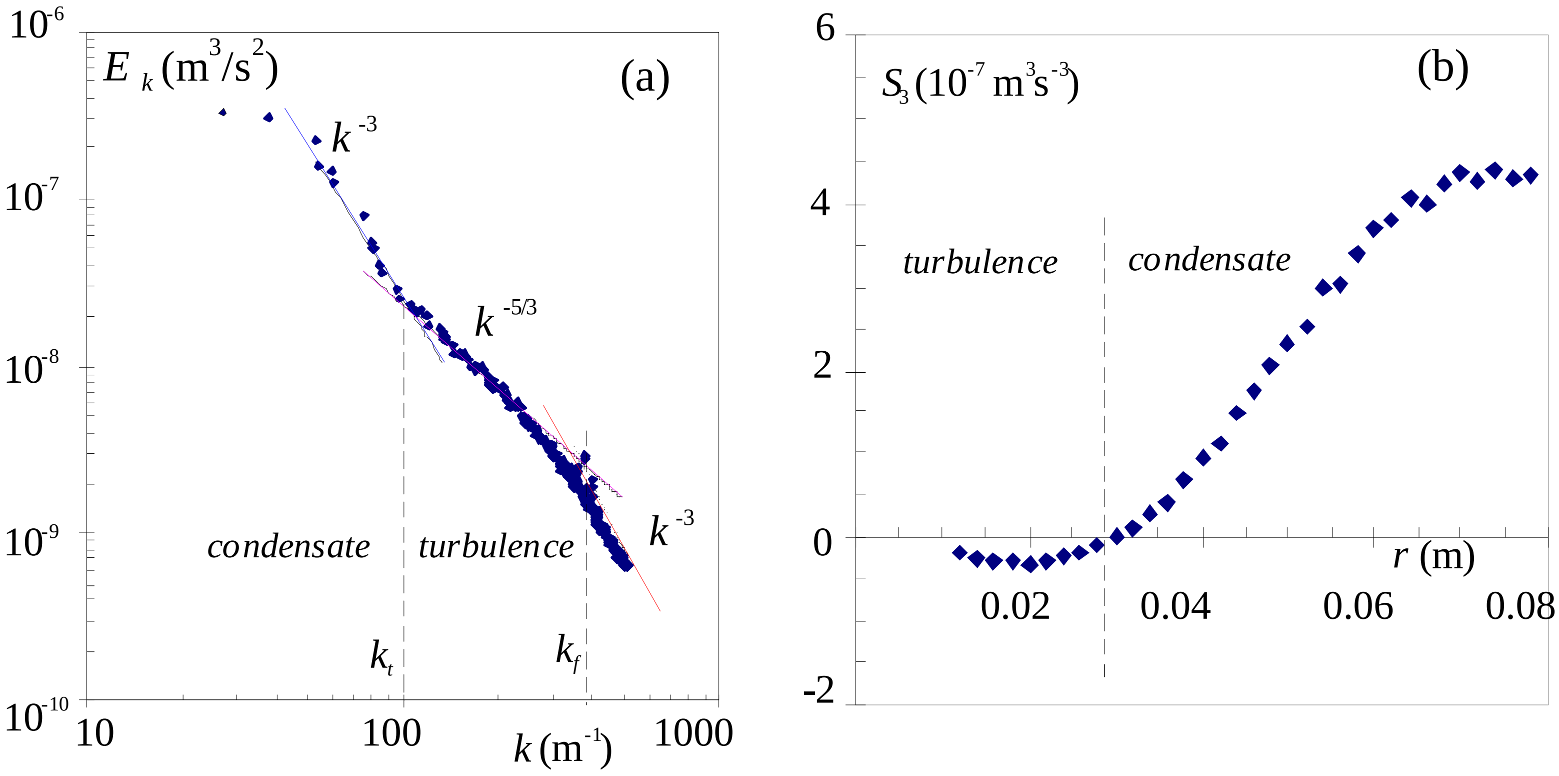


Lab experiment, weak spectral condensate

Shats et al, PRL2007



Case of strong condensate



$$\delta V = \delta \overline{V} + \delta \tilde{V}$$

Mean shear flow (condensate) changes all velocity moments:

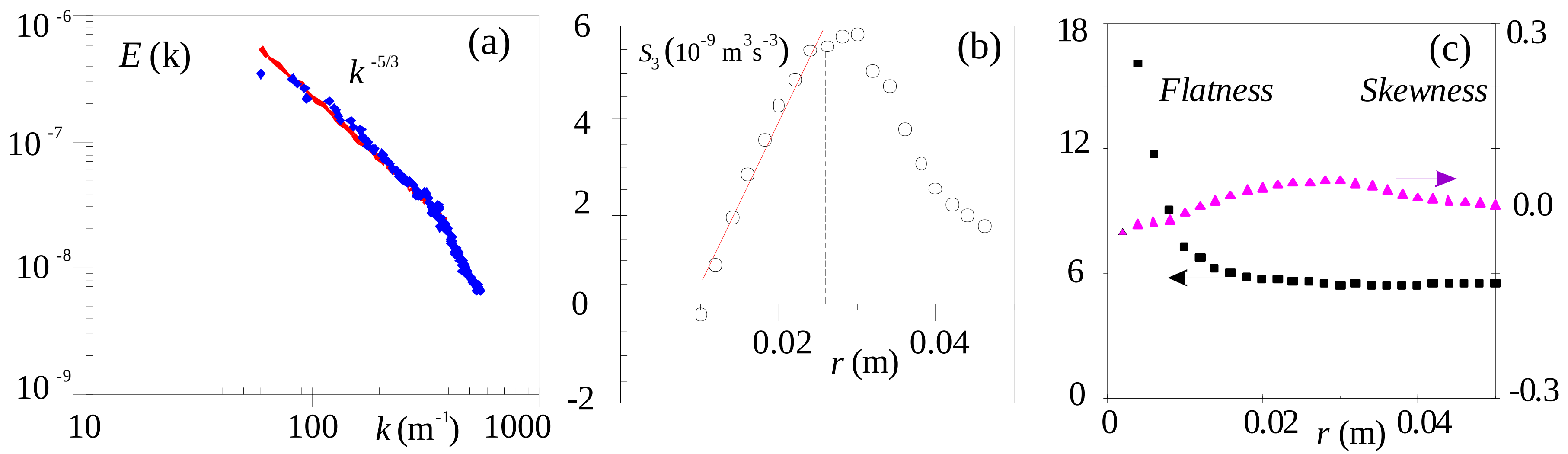
$$\left\langle \delta V^2 \right\rangle = \left\langle \delta \overline{V} + 2\delta \overline{V}\delta \tilde{V} + \delta \tilde{V}^2 \right\rangle$$

$$\left\langle \delta V^3 \right\rangle = \left\langle \delta \overline{V}^3 - 3\delta \overline{V}^2 \delta \tilde{V} + 3\delta \overline{V}\delta \tilde{V}^2 - \delta \tilde{V}^3 \right\rangle$$

Mean subtraction recovers isotropic turbulence

1.Compute time-average velocity field (*N*=400): $\overline{V}(x,y) = 1/N \sum_{n=1}^{N} V(x,y,t_n)$

2. Subtract $\overline{V}(x,y)$ from *N*=400 instantaneous velocity fields



Recover ~ $k^{-5/3}$ spectrum in the energy range

Kolmogorov law – linear S3 (r) dependence in the "turbulence range";

Kolmogorov constant C≈7

Skewness *Sk* ≈ 0 , flatness slightly higher, *F* ≈ 6

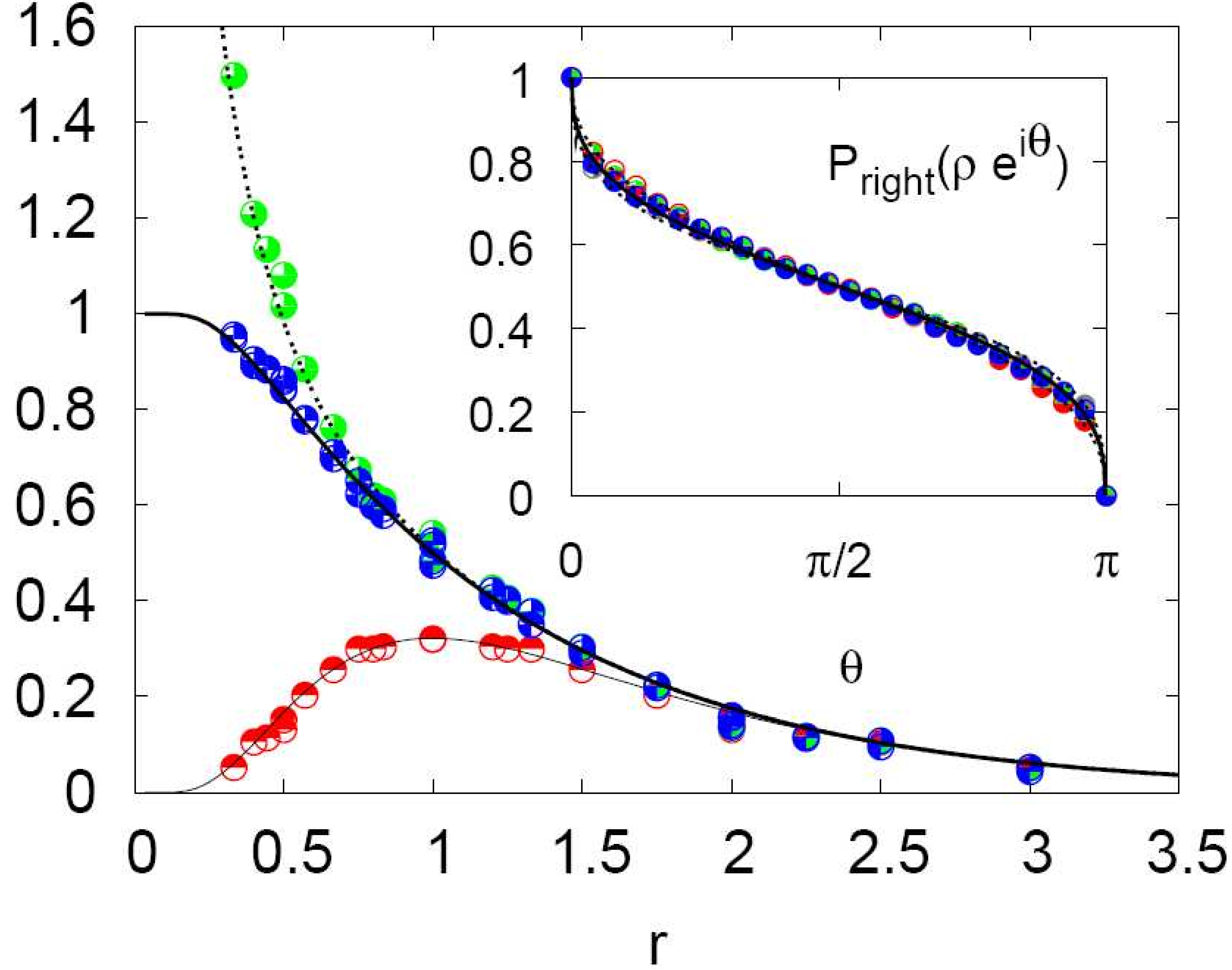
Conclusion

Inverse cascades seems to be scale invariant.

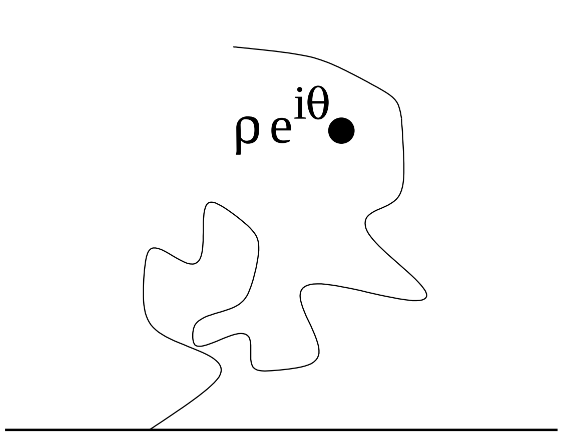
Within experimental accuracy, isolines of advected quantities are conformal invariant (SLE) in turbulent inverse cascades.

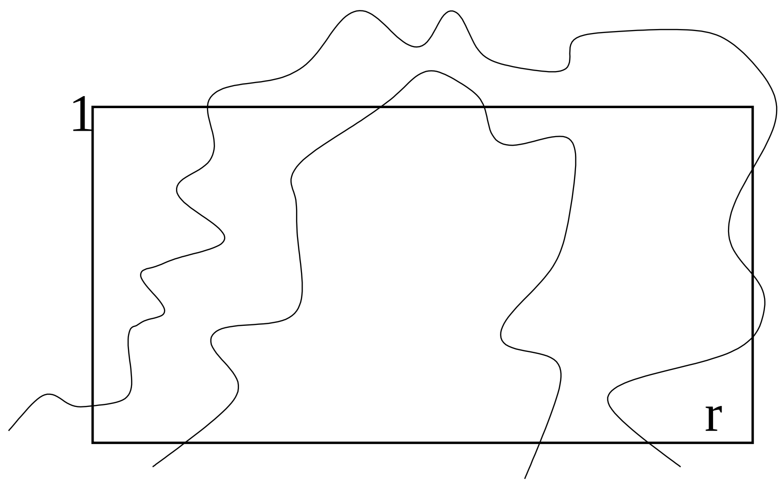
Condensation into a system-size coherent mode breaks symmetries of inverse cascades.

For Gross-Pitaevsky equation, condensate may make turbulence conformal invariant



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
r
θ
$P_{right}(\rho\, e^{i\theta})$
1
0.8
0.6
0.4
0.2
0
0
π/2
π



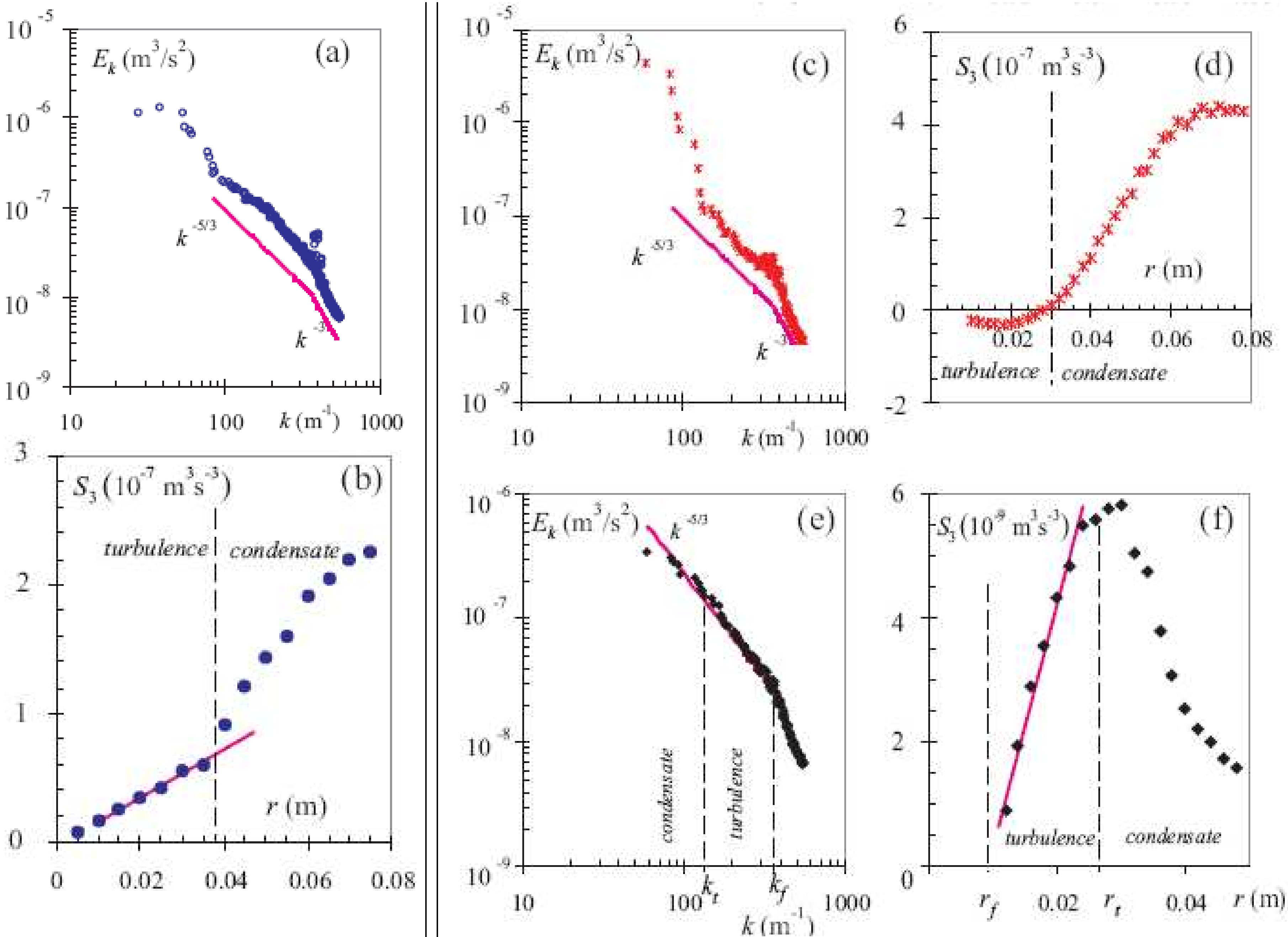


$$P = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})}\ {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -\cot^2\theta\right)\cot\theta$$

$$\pi_v = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}\eta^{1/3}\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \eta\right) \text{ with } \eta = [(1-k)/(1+k)]^2$$

$$r = K(1-k^2)/[2K(k^2)]$$

$$\pi_{hv} = \pi_v - \frac{\eta}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}\ {}_3F_2\left(1, 1, \frac{4}{3}; 2, \frac{5}{3}; \eta\right)$$



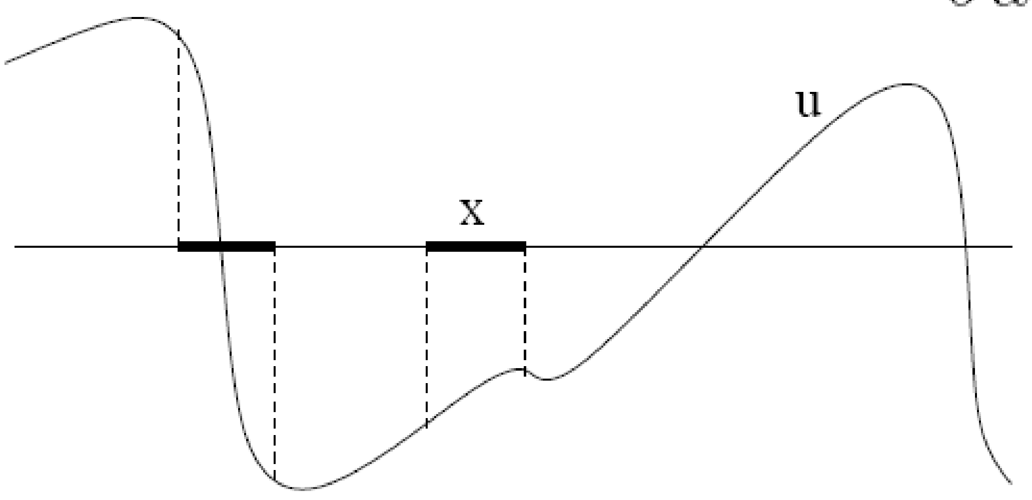
Weak condensate

Strong condensate

Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0$$

$$\delta u(x,t) = u(x,t) - u(0,t)$$

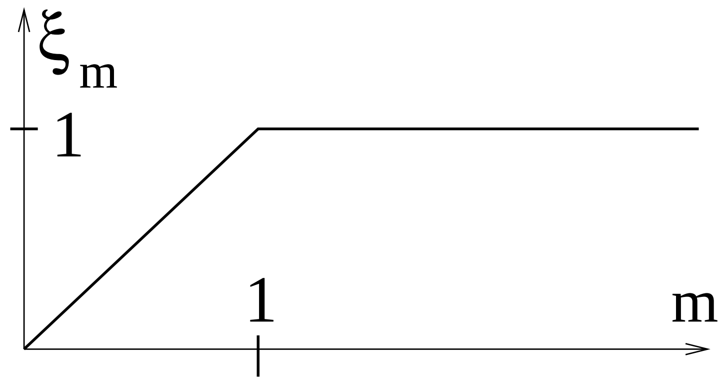


$$S_n = \langle \delta u^n \rangle = C_n x^{\xi_n}$$

$$S_n(x) \sim C_n|x|^n + C'_n|x|$$

Typical velocity profile in Burgers turbulence

$$P(\delta u, r) \neq (\delta u)^{-1} g(\delta u / r^a)$$



Anomalous scaling in terms of the integrals of motion

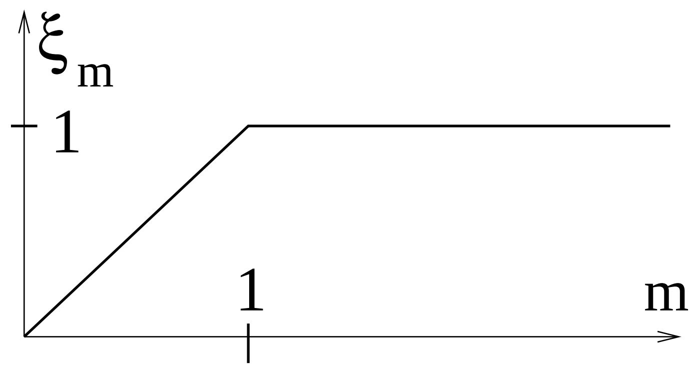
$$u_t + uu_x - \nu u_{xx} = 0$$

$$E_n = \int u^{2n}\, dx/2$$

$$\langle \dot{E}_n \rangle = \epsilon_n$$

$$S_{2n+1} = -4(2n+1)\epsilon_n x/(2n-1)$$

$$S_n = \langle \delta u^n \rangle = C_n x^{\xi_n}$$



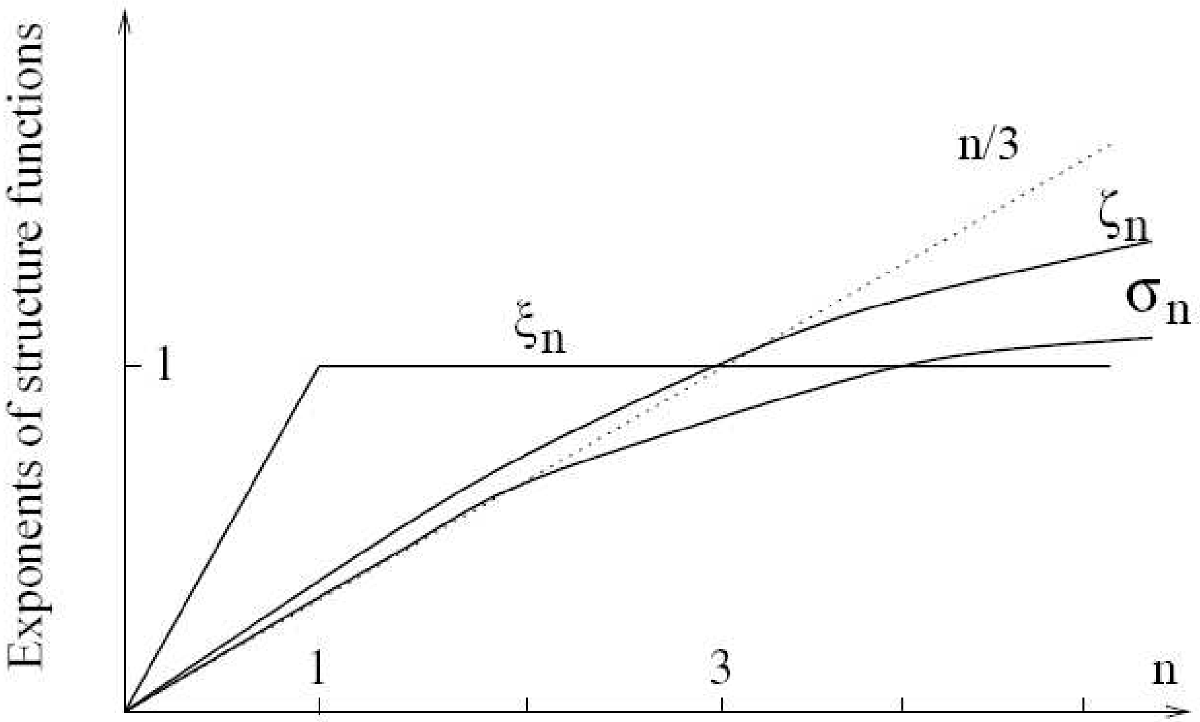


Fig. 1.4. The scaling exponents of the structure functions ξ_n for Burgers, ζ_n for 3d Navier-Stokes and σ_n for the passive scalar. The dotted straight line is $n/3$.

Statistical conservation laws

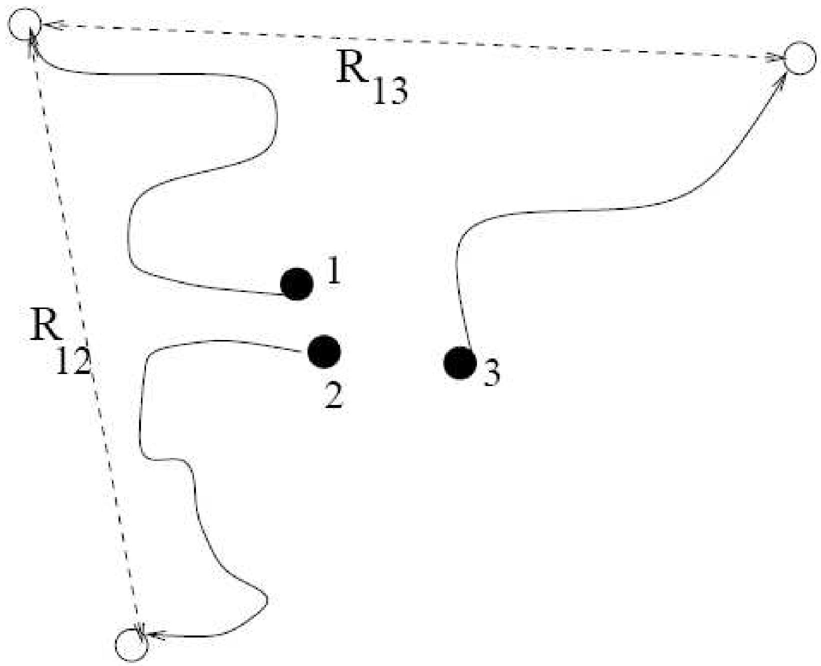


Fig. 1.5. Three fluid particles in a flow.

$$< R_{ij}^2 > \propto t^a$$

$$f(\lambda R_{ij}) = \lambda^\zeta f(R_{ij})$$

$$f \propto t^{a\zeta/2}$$

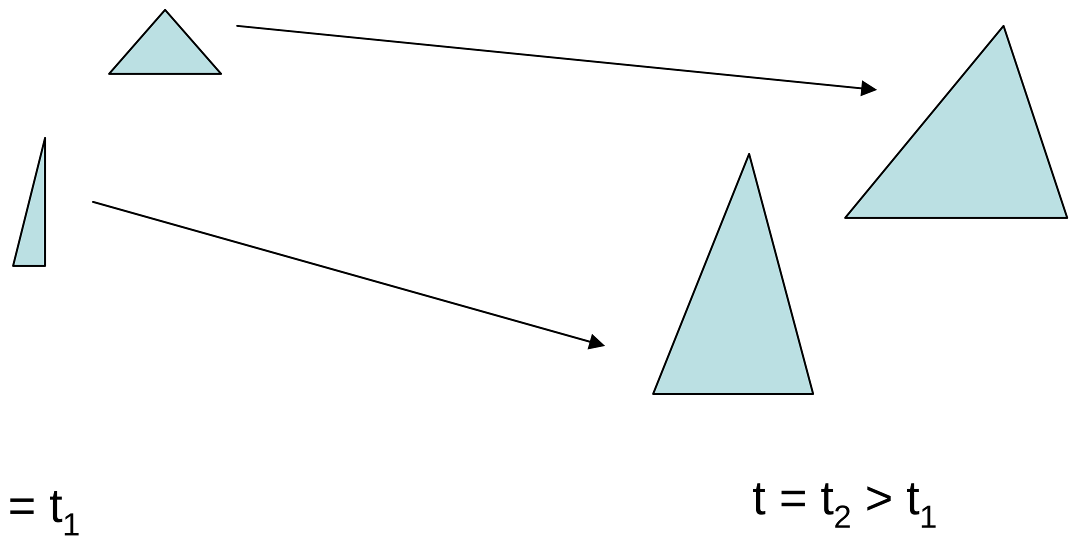
for Brownian walk

$$f_2 = \langle R_{12}^2 - R_{34}^2 \rangle = \langle R^2 \cos 2\theta \rangle$$

$$\theta = \arcsin(R_{12}/R)$$

$$R^2 = R_{12}^2 + R_{13}^2$$

$$f_4 = \langle 2(d+2) R_{12}^2 R_{34}^2 - d(R_{12}^4 + R_{34}^4) \rangle = \langle R^4 [(d+1)\cos^2 2\theta - 1] \rangle$$



Box 2: Growth in size and reduction of shape fluctuations. Schematic of two configurations of three particles at the apex of triangles to the left, as they move about in a turbulent flow. The sizes of the ``clouds'' increase with time but the fluctuations in their shape decrease.

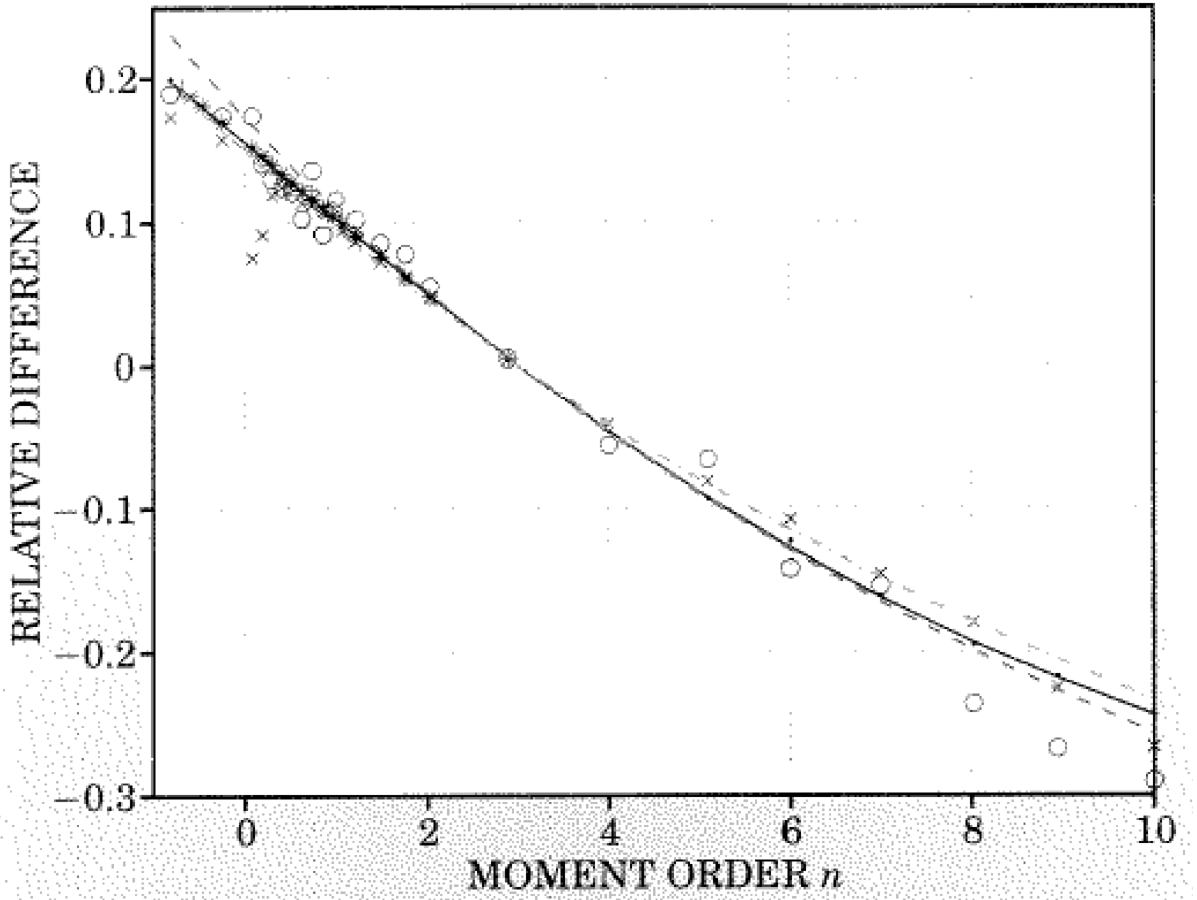
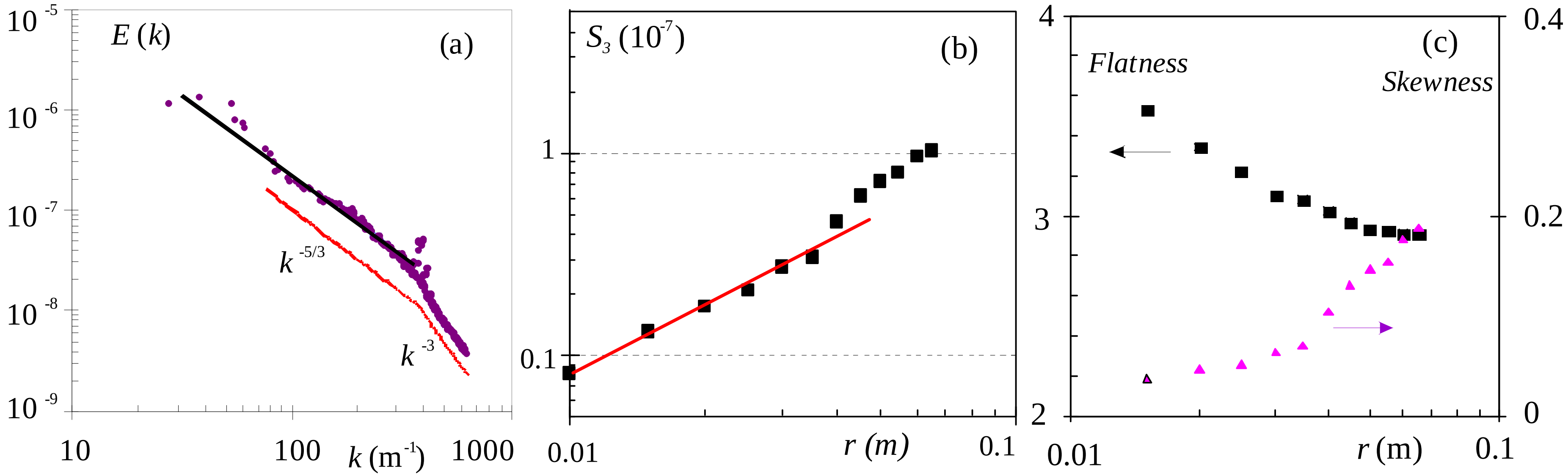


Figure 3. Scaling exponents ζ_n differ from the scale-invariant, linear Kolmogorov value of $n/3$. In this plot, the relative difference is defined as $(\zeta_n - n/3)/(n/3)$, open circles show experimental results, crosses and stars show results of simulations, and the curves give various theoretical results. Note that the relative difference passes through $n = 0$ without showing any special feature. Nor does there appear to be any special behavior as n approaches -1, notwithstanding that structure functions of order -1 and lower are undefined. (Adapted from ref. 3.)

Case of weak condensate

$$S_{3L}(r) = \left\langle \delta V_L^3 \right\rangle + \left\langle \delta V_L \delta V_T^2 \right\rangle = \frac{3}{2}\varepsilon r$$

$$Sk = S_3 / (S_2)^{3/2}$$
$$F = S_4 / S_2^2$$



Weak condensate case shows small differences with isotropic 2D turbulence

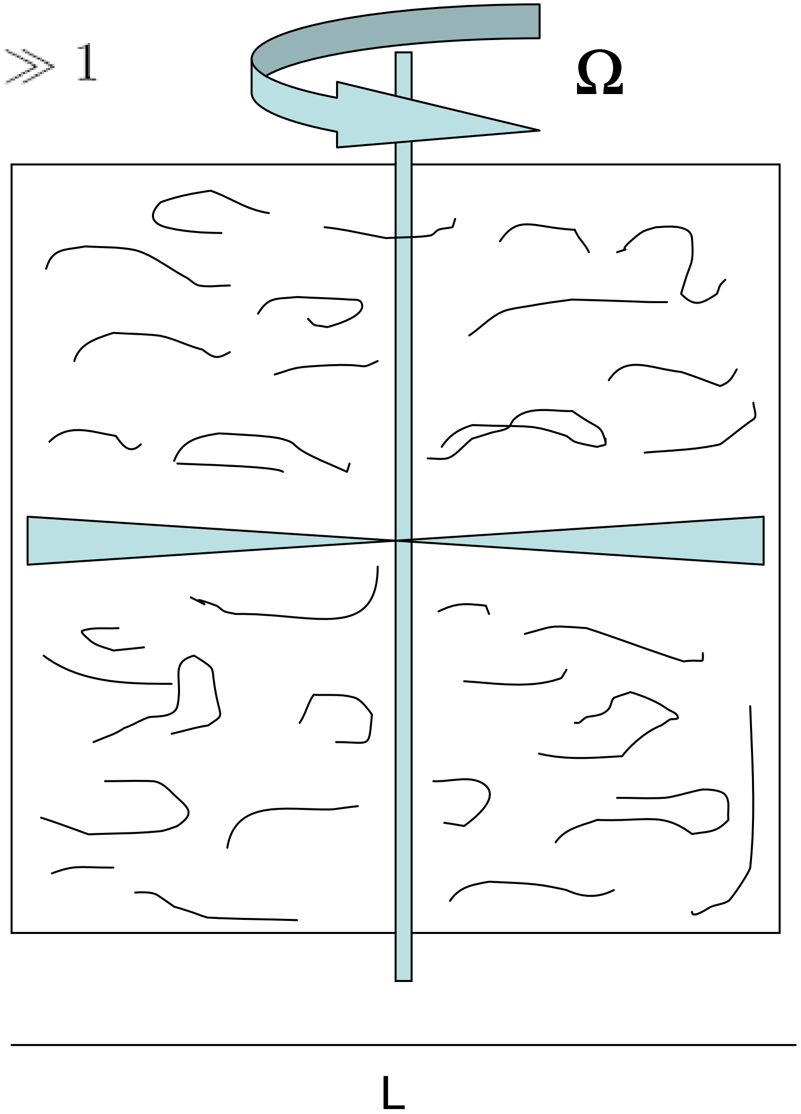
~ $k^{-5/3}$ spectrum in the energy range

Kolmogorov law – linear S3 (r) dependence; Kolmogorov constant C≈5.6

Skewness and flatness are close to their Gaussian values (*Sk*=0, *F*=3)

Turbulence is a state of a physical system with many degrees of freedom deviated far from equilibrium. It is irregular both in time and in space.

$$Re = VL/\nu = \Omega L^2/\nu \gg 1$$



Physics Today **59**(4), 43 (2006)