MHV amplitudes in N=4 SUSY Yang-Mills theory and quantum geometry of the momentum space

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- QCD is well defined theory of strong interactions. However the vacuum structure is not clarified. It behaves perturbatively at high energy because of the asymptotic freedom

$$g^2_{YM}(q^2) \propto \frac{1}{log(q^2/\Lambda^2_{QCD})}$$

. Summation of the perturbative series in the coupling constant is questionable. No explicit examples, just a few first terms are known for any physically interesting objects.
- The QCD is expected to have a stringy description. Indications; High energy amplitudes have the Regge behavior consistent with the dual Veneziano formula. The spectrum of hadrons enjoys the description in terms of the Regge trajectories 1960-1970
- What is the string theory behind QCD? How the tension of the string depends on the coupling constant? e.t.c. It seems

- Great progress during the last 10 years - gauge/string duality! Polyakov-Maldacena
- The strong limitation- duality is valid for the gauge theory with the maximal N=4 SUSY in four dimensions. This gauge theory is dual to the string propagating in ten dimensional space $AdS_5 \times S^5$.
- The tension of the string is proportional to $T \propto \sqrt{g_{ym}^2 N_c}$. Hence at strong coupling the tension is large and the string is quasiclassical. A lot of results at strong coupling were derived! However at weak coupling were the direct comparison with gauge theory calculations is possible the stringy sigma model is in a deep quantum regime. Needs for the proper formulation.

- The simplest object to compare the field theory calculations and the string prediction - anomalous dimensions of the conformal operators. They correspond to the energy of the string. Operators with the large anomalous dimensions – large energy of the string– quasiclassics.
- Example; $\Phi D^S \Phi$ where D-covariant derivative and S-Lorentz spin of the operator. The anomalous dimension of this operator at large S behaves as $\gamma_s = f(g_{YM}) logS$
- There is all-loop answer for this anomalous dimensions!(Beisett-Eden-Staudacher)! It agrees with all gauge theory calculations up to four loops and with strong coupling calculations via strings. This is the first example of the all-loop answer!

- Much more interesting object - scattering amplitude. There are many cancelations due to SUSY but they are still highly complicated. MHV amplitudes (++——) are the simplest objects depending on a lot of kinematical invariants to discuss within the gauge/string duality
- Simplification at large N_c - MHV amplitudes are described by the single function of the kinematical variables. The amplitudes have Regge limit - place to search for the strings. Properties of the tree MHV amplitudes
- Holomorphy - it depends only on the "'half" ' of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$(Parke-Tailor, 87)
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions of the sphere

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Twistor space - $CP(3\|4)$
- Twistor space emerges if we make a Fourier transform with respect to the "half" of the momentum variables $\int d\lambda e^{i\mu\lambda} f(\lambda\bar{\lambda})$. Point in the Minkowski space corresponds to the plane in the twistor space
- Localization follows from the holomorphic property of the tree MHV amplitude. Possible link to integrability via fermionic representation
- Stringy interpretation - auxiliary fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.

- The tree MHV amplitude has very simple form

$$A(1^-, 2^-, 3^+ \dots, n^+) = g^{n-2} \frac{< 12 >^4}{< 12 >< 23 > \cdots < n1 >}$$

- The on-shell momenta of massless particle in the standard spinor notations read as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors.
- Inner products in spinor notations $< \lambda_1, \lambda_2 >= \epsilon_{ab} \lambda_1^a \lambda_2^b$ and $[\tilde{\lambda}_1 \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$.

- The generating function for the tree MHV amplitudes - solution to the self-duality equation with the particular boundary conditions (Bardeen 96, Rosly-Selivanov 97). It substitutes the sum of the plane waves in the nonlinear theory

$$A_{\dot\alpha} = g^{-1}\partial_{\dot\alpha} g$$

$$g_{ptb}(\rho) = 1 + \sum_J g_J(\rho) E_J + \cdots + \sum_{J_1 \dots J_L} g_{J_1 \dots J_L}(\rho) E_{J_1} \dots E_{J_L} + \dots$$

- The E_{J_1} is the solution to the free equation of motion
- The coefficients are derived from self-duality condition

$$g_{J_1 \dots J_L}(\rho) = \frac{< \rho, q^{j_1} >< j_1, q^{j_2} >< j_2, q^{j_3} > \cdots < j_{L-1}, q^{j_L} >}{< \rho, j_1 >< j_1, j_2 >< j_2, j_3 > \cdots < j_{L-1}, j_L >}$$

- The resummation of the tree amplitudes can be done resulting into the so-called MHV Lagrangian (Cachazo-Scwrcek-Witten). The tree MHV amplitude corresponds to the vertex in this formulation
- The same solution to the self-duality equation provides the canonical transformation from the tree light-cone YM Lagrangian to the MHV Lagrangian (Rosly-A.G., 04).
- The proper analogy: instanton solution to the selfduality equation generates t'Hooft vertex in QCD. Here the different solution to the selfduality equation (perturbiner) generates the infinite set of MHV vertexes

Properties of the loop MHV amplitudes

- Exponentiation of the ratio $\frac{M_{all-loop}}{M_{tree}}$ which contains the IR divergent and finite parts.
- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

- It involves only two main ingredients - one-loop amplitude and all-loop $\Gamma_{cusp}(\lambda)$
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively
- The conjecture fails starting from six external legs at two loops (Bern -Dixon-Kosower, Drummond-Henn-Korchemsky-Sokachev, Lipatov-Kotikov) and at large number of legs at strong coupling(Alday-Maldacena)

- One more all-loop conjecture - $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Abelian Wilson polygon built from the external light-like momenta p_i.
- The conjecture was formulated at strong coupling (Alday-Maldacena, 06) upon the T-duality at the worldsheet of the string in the AdS_5 geometry
- Checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- Important role of Ward identities with respect to the special conformal transformation in determination of the Wilson polygon (Drummond-Henn-Korchemsky-Sokachev)
- There is no satisfactory stringy explanation of the loop MHV amplitudes and this duality. Suspicion - closed string modes contribute (Cachazo-Swrchek-Witten)that is perturbative

Main Questions

- Is there fermionic representation of the loop MHV amplitudes similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability behind the amplitudes is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- Is there trace of the weak-strong coupling S-suality of N=4 SYM in the amplitudes?
- What is the stringy geometrical origin of the BDS conjecture, if any?
- What is the physical origin of MHV amplitude-Wilson polygon duality?

c=1 example

- Consider c=1 string (1d-target space + Liouville direction). The only degrees of freedom - massless tachyons with the discrete momenta
- Exact answer for the tachyonic amplitudes (Dijkgraaf,Plesser,Moore 94)
- Generating function for the amplitude - solution (τ function) for the Toda integrable systems. "'Times"'- generating parameters for the tachyon operators with the different momenta

- Generating function admits representation via chiral fermions or bosons on the Riemann surface=Fermi surface for the auxiliary fermions
$$x^2 - y^2 = 1$$
in the background of the particular abelian gauge field A(z) which provides the "S-matrix"
- This Riemann surface parameterizes the particular moduli space.
- The "fermions" represent the intersection of noncompact Lagrangian branes (they cover the half of the whole dimension) so-called FZZT branes. They are not literally fermions - better to think of as Wigner functions on the phase space. Two types of branes ZZ branes - localized in the Lioville direction but extended on the Riemann surface. FZZT branes- extended(semiinfinitely) in the Lioville direction and

- The generating function for the amplitude

$$\tau(t_k) =< 0|exp(\sum t_k V_k)exp\int(\bar{\psi}A\psi)exp(\sum t_{-k}V_{-k})|0>$$

- That is all scattering amplitudes can be described in terms of the fermionic currents on the Fermi surface

- The amplitude can be represented in terms of the "'Wilson polygon"' for the auxiliary abelian gauge field! This gauge field has nothing to do with the initial tachyonic scalar degrees of freedom. The auxiliary abelian gauge field A(z) yields the choice of the vacuum state in the theory of fermions.
- Riemann Fermi surface reflects the hidden moduli space of the theory (chiral ring) and it gets quantized. Equation of the Riemann surface becomes the operator acting on the wave function(the analogue of the secondary quantization). The following commutation relation is implied

$$[x, y] = i\hbar$$

- This procedure of the quantization of the Riemann surface is familiar in the theory of integrable systems. Quantum Riemann surface =so-called Baxter eqution
- Degrees of freedom on the Riemann surface - Kodaira-Spencer gravity reduced to two dimensions (Dijkgraaf-Vafa,07)
- Solution to the Baxter equation - wave function of the single separated variable - Lagrangian brane or Lagrangian branes intersection (Nekrasov-Rubtsov-A.G. 2000)
- Polynomial solution to the Baxter equation - Bethe equations for the roots

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. Inequivalent triangulations of the surface can be mapped into set of geodesics on the upper half-plane
- This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- The natural objects geodesics can be determined in terms of shear variables z_a
- The simplectic structure in terms of these variables is simple $\sum_a dz_a \wedge dz_b$ where a corresponds to oriented edge and b is edge next to the right
- Upon quantization

$$[Z_a, Z_b] = 2\pi\hbar\{z_a, z_b\}$$

- Quantum mechanically there is operator of the "'duality"' K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "'quantum spectral curve operator"' Classically this curve looks as

$$e^u + e^v + 1 = 0$$

and gets transformed into the Baxter equation

$$(e^{i\hbar\partial_v} + e^v + 1)Q(v) = 0$$

with the Poisson bracket

$$[v, u] = i\hbar$$

- Let us use the representation for the finite part of the one-loop amplitude as the sum of the following dilogariphms. The whole amplitude is expressed in terms of the sums of the so-called two easy-mass box functions

$$\sum_i \sum_r Li_2\left(1 - \frac{x_{i,i+r}^2 x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2}\right)$$

$$x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta of gluons

- One-loop amplitude with n-gluons is described in terms of the "fermions" living on the spectral curve=Fermi surface which is embedded into the four dimensional complex space! MHV loop amplitude - fermionic current correlator on the spectral curve. Fermi surface parameterizes the moduli space $M_{0,4}$
- BDS conjecture for all-loop answer=quasiclassics of the fermionic correlator with the identification

$$\hbar^{-1} = \Gamma_{cusp}(\lambda)$$

- "Fermions" on Fermi surface represent the intersection of the noncompact Lagrangian branes in the B model along the Riemann surface. In the mirror dual geometry fermions represent Lagrangian branes. Degrees of freedom on the intersections live on the moduli space of the complex structures. They are transformed nontrivially on the Fermi surface on the surface because of its quantum nature
- Geometry: The spectral curve is embedded as the holomorphic surface in the internal 4-dimensional complex space

$$xy = e^u + e^v + 1$$

- Two Lagrangian 2-planes in C^4 have the geometry

$$x = 0 \quad e^u + e^v + 1 = 0$$

and

$$y = 0 \quad e^u + e^v + 1 = 0$$

- Classically we have degrees of freedom on the intersection of the Lagrangian branes. There are also open strings, representing gluons with the disk geometry ending on the pair of Lagrangian branes. These strings correspond to the gluons.
- The total space involved is approximately C^8. The factor C^4 is the complexified Minkowski space and the second factor C^4 corresponds to the twistor space in the realization of the complexified Minkowski space as $Gr(2,4)$. That is we have coordinate space, momentum space and the additional C^2 responsible for the polarization of the phase space. No need for the additional dimensions not related with our conventional coordinates
- The space where the string propagates is essentially noncommutative because of the conventional Planck constant. This is essential when the loop effects in the gauge

- We expect weak-strong coupling duality valid in N=4 theory. D3 Lagrangian branes are self-dual but F1 strings gets substituted dy D1 strings that is the two-dimensional YM theory with the marked points on their worldvolume is relevant
- The origin of the Riemann surface. It corresponds to the summation of all anomalous relations in the gauge theory. Nontrivial effect of regulator degrees of freedom.
- Similar emergence of the Riemann surfaces. N=2 SYM theory-surface follows from the summation of the infinite number of the instantons. N=1 SYM- the surface is the result of the account of all generalized Konishi anomalies under the transformations $\Phi \to F(\Phi)$.

- Quantization of the Fermi surface involves the YM coupling constant
$$\frac{1}{g_{YM}^2} = \frac{\int B_{NS-NS}}{g_s}$$
Usually it is assumed that g_s yields the "Planck constant" for the quantization of the moduli space of the complex structures in the Kodaira-Spenser gravity. However equally some function of Yang-Mills coupling can be considered as the quantization parameter.
- The YM coupling constant yields the quantization of the gravity degrees of freedom in the box diagram (light-on-light scattering)

- Quasiclassics for the solution to the equation of the quantized Fermi surface

$$\Psi(z,\hbar) = \int \frac{e^{ipz}}{p \times sinh(\pi p) sinh(\pi \hbar p)} dp$$

reduces to

$$\Psi(x) \to exp(\hbar^{-1} Li_2(x) + \ldots)$$

- Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.
- The quantum dilogariphm has the dual-symmetric form

$$\Psi(z,\hbar) = \frac{e_q(\omega)}{e_{\tilde{q}}(\tilde{\omega})}$$

$q\tilde{q} = 1$ and $e_q(z) = \prod(1 - zq^n)$

- It can be vizualized as two "left" and "right" lattices

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|J(z_1)...J(z_n)exp(\psi_k A_{nk}\psi_n)|0 >$$

- The variables ψ_k are the modes of the fermion on the spectral curve and $J(z)$ is the fermionic current. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

- In the Regge limit the operator of the quantum Fermi surface plays the important role; it provides the Lipatov,s duality transformation between the coordinates and momenta.
- From the worldsheet viewpoint one considers the discretization of the Liuville mode and the Faddeev-Volkov model yields the good candidate for the correct S-matrix. In the target space the natural integrable system is described by the model with the universal R-matrix based on the modular double

$$D = U_q(SL(2,R)) \otimes U_{\tilde{q}}(SL(2,R))$$

- The universal R-matrix for the modular double acting in $D \otimes D$ involves the product of four quantum dilogs desired to describe the "basic block"-box diagram
- Natural link with the spin chains for the Regge limit when the Wilson polygon becomes very thin in one direction

Conclusion

- The representation of the loop MHV amplitude as the chiral correlator on the quantized Fermi surface of the effective degrees of freedom is suggested. Nontrivial effect of closed string degrees of freedom(Kodaira-Spencer gravity) in the box diagram!
- The natural issue of the S-duality in the B-model description. Expected modular properties of the amplitudes at the generic couplings. Integrability is responsible for the selfconsistency of the whole picture
- Link to the integrability behind generic MHV amplitudes via fermionic representation. Particular solutions to 3-KP integrable system which correspond to the Faddeev-Volkov model of the discrete conformal mappings with the good S-duality properties. The corresponding statistical model with

- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space with $\Gamma_{cusp}(\lambda)$ as the quantization parameter. Way to improve-take into account the cubic vertex (screening operator) on the world-sheet in the Kodaira-Spencer gravity. Hopefully this improves the matching with the Regge limit of the amplitudes lost in BDS anzatz
- Wilson polygon - MHV amplitude duality is based on the fermionic representation of the amplitude and the gauge field is the "Berry connection"

- The degrees of freedom responsible for the dual description of the gluon amplitudes - branes=hypersurfaces in the momentum space.
- More generally it is possible to consider the intersection of Lagrangian branes in the C^8 involving both phase space and twistor parts. That is one could thought about M5 branes as Lagrangian submanifolds. The orientation of the Lagrangian brane is fixed dynamically. One example M5 brane description of N=2 SYM theory (Witten)upon the summation over the instantons in N=2 theory (Nekrasov). Another example - description of the amplitudes.
- Some strange anomaly has been found in the box diagram long time ago (Ioffe,Khodjamirian,A.G -89). It turns out that just this object provides the meeting of the gauge and gravity degrees of freedom