Universal and non-universal tails of distribution functions in the directed polymer and Kardar-Parisi-Zhang problems

I.V. Kolokolov and S.E. Korshunov

L.D. Landau Institute for Theoretical Physics

- Relation between different models
- Optimal fluctuation approach
- Far-left and far-right tails
- Inclusion of the renormalization effects
- Most close (universal) tails

Phys. Rev. B **75**, 140201 (2007)
arXiv: 0805.0402 (2008)

THE DIRECTED POLYMER PROBLEM

An elastic string in a random potential:

$$H = \int_0^L dt \left\{ \frac{J}{2} \left[\frac{d\mathbf{x}(t)}{dt} \right]^2 + V[t, \mathbf{x}(t)] \right\}$$

Gaussian distribution with

$$\overline{V(t,\mathbf{x})} = 0, \quad \overline{V(t,\mathbf{x}))V(t',\mathbf{x}')} = \delta(t-t')U(\mathbf{x}-\mathbf{x}')\,.$$

We are interested in probability of large fluctuations of the polymer free energy F, that is in the tails of the distribution function

$$P_L(F) \equiv \exp[-S(F)]$$

(1) for very large deviations of F from its average;
(2) in the universal regime, where $P_L(F) = \mathcal{P}[F/F_*(L)]$.

The transverse dimension will be denoted d.

THE RELATION TO KPZ and BURGERS PROBLEMS

Huse, Henley and Fisher (1985)

The evolution of the polymer's partition function:

$$z(t,\mathbf{x}) = \int d\mathbf{x}'\, z(0,\mathbf{x}') \int_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(t)=\mathbf{x}} \mathcal{D}\mathbf{x}(t') \exp\left(-H/T\right)$$

is governed by an imaginary-time Schrödinger equation:

$$-T\frac{\partial}{\partial t} z(t,\mathbf{x}) = \left[-\frac{T^2}{2J}\nabla^2 + V(t,\mathbf{x})\right] z(t,\mathbf{x})\,;$$

of the free energy $f(t,\mathbf{x}) = -T \ln z(t,\mathbf{x})$ – by the KPZ equation:

$$\frac{\partial f}{\partial t} + (1/2J)(\nabla f)^2 - \nu\nabla^2 f = V(t,\mathbf{x})\,;$$

and of $\mathbf{u}(t,\mathbf{x}) \equiv \nabla f/J$ – by a randomly stirred Burgers equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla \mathbf{u}^2 - \nu\nabla^2 \mathbf{u} = \frac{1}{J}\nabla V(t,\mathbf{x})$$

THE BOUNDARY CONDITIONS

We will be interested in fluctuations of $F \equiv f(L, \mathbf{x} = 0)$, which in terms of the DP problem means that at $t = L$ the polymer is fixed at $\mathbf{x} = 0$.

If at the other end (at $t = 0$) the polymer is not fixed, the initial condition for $f(t, \mathbf{x})$ is $f(0, \mathbf{x}) = \text{const}$ - the standard initial condition for the non-stationary KPZ problem.

If at $t = 0$ the polymer is also fixed, that is $\mathbf{x}(t = 0) = 0$, this means that at $t \to 0$

$$\mathbf{u}(t, \mathbf{x}) \to \frac{\mathbf{x}}{t}\,.$$

THE OPTIMAL FLUCTUATION APPROACH

The probability of any disorder realization $V(t,\mathbf{x})$ is determined by the functional $S\{V\} = \frac{1}{2}\int_0^L dt \int d\mathbf{x} \int d\mathbf{x}' V(t,\mathbf{x}) U^{-1}(\mathbf{x}-\mathbf{x}') V(t,\mathbf{x}')$ or

$$S\{f,\mu\} = \int_0^L dt \left\{ \int d\mathbf{x}\, \mu(t,\mathbf{x}) \left[\frac{\partial f}{\partial t} + \frac{(\nabla f)^2}{2J} - \nu \nabla^2 f \right] - \frac{1}{2} \int \int dx\, dx' \mu(t,\mathbf{x}) U(\mathbf{x}-\mathbf{x}') \mu(t,\mathbf{x}') \right\} .$$

The optimal fluctuation approach assumes that the probability of a rare fluctuation is dominated by a particular disorder realization which minimizes S for the given boundary conditions.

The above expression for $S\{f,\mu\}$ has extremum when

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{(\nabla f)^2}{2J} - \nu \nabla^2 f &= V(t,\mathbf{x}) , \\ \frac{\partial \mu}{\partial t} + \mathrm{div}(\mathbf{u}\mu) + \nu \nabla^2 \mu &= 0 \end{aligned}$$

where $V(t,\mathbf{x}) = \int d\mathbf{x}'\, U(\mathbf{x}-\mathbf{x}') \mu(t,\mathbf{x}')$.

THE FAR-LEFT TAIL

Here one can start by looking at the "stationary" solution with $V(t,\mathbf{x}) \equiv V(\mathbf{x})$. This reduces the equation for $z(t,\mathbf{x}) \equiv \exp(-E_0 t/T)\Psi(\mathbf{x})$ to

$$E_0\Psi(\mathbf{x}) \;=\; \left[-\frac{T^2}{2J}\nabla^2 + V(\mathbf{x})\right]\Psi(\mathbf{x})\,.$$

Simultaneous minimization of E_0 and

$$S = \frac{L}{2}\int\int d\mathbf{x}\,d\mathbf{x}'\,V(\mathbf{x})[U(\mathbf{x}-\mathbf{x}')]^{-1}V(\mathbf{x}')\,.$$

leads (Halperin and Lax, 1966, I.M. Lifshitz, 1966) to

$$E_0\Psi = -(T^2/2J)\nabla_x^2\Psi - \Lambda\int dx'\,\Psi(x)U(x-x')\Psi^2(x')\,,$$

that is

$$V(\mathbf{x}) = -\Lambda\int d\mathbf{x}'\,U(\mathbf{x}-\mathbf{x}')\Psi^2(\mathbf{x}')$$

δ-functional correlations: $U(\mathbf{x}-\mathbf{x}') = U\delta(\mathbf{x}-\mathbf{x}')$

$d = 1$ - exactly solvable case

$$V(x) = -\frac{T^2}{J\Delta^2 \cosh^2(x/\Delta)} \qquad \text{with} \quad \Delta = T\left[\frac{L}{2J(-F)}\right]^{1/2}$$

from where

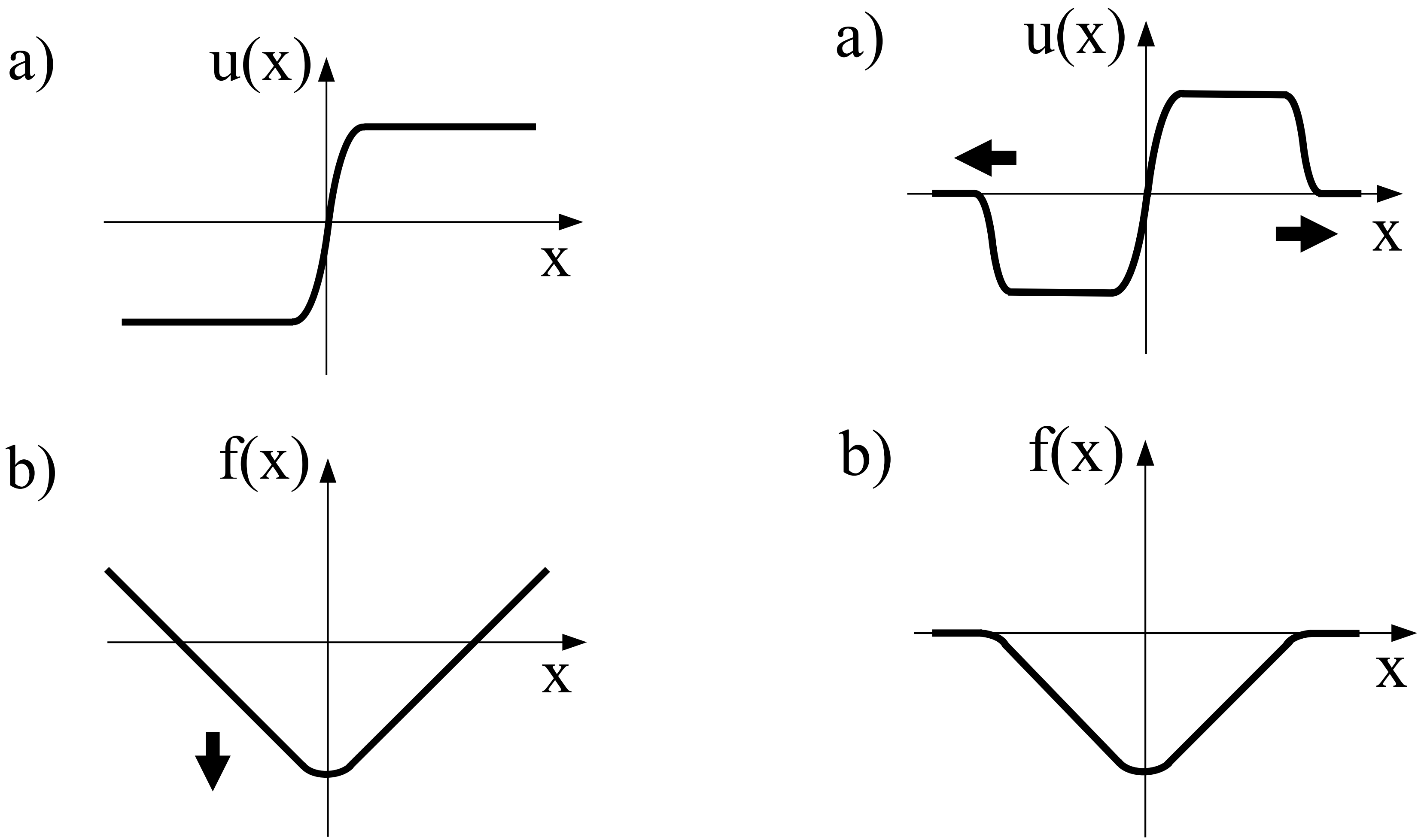
$$S = \frac{4}{3}\frac{T}{U_0}\left[\frac{2(-F)^3}{JL}\right]^{1/2}$$

$0 < d < 4$:

Both kinetic energy $\mathcal{K} \sim T^2/J\Delta^2$ and absolute value of potential energy $\mathcal{V} \sim V(0)$ have to be of the same order as $E_0 \equiv F/L$. Taking into account that S can be estimated as $L\Delta^d[V(0)]^2/U_0$, one gets

$$S(F) \sim \frac{T^d(-F)^{2-d/2}}{U_0 J^{d/2} L^{1-d/2}}\,.$$

THE NON-STATIONARY MODIFICATION OF THE STATIONARY SOLUTION $(d = 1)$



At $d > 2$ the expression we have obtained

$$S(F) \sim \frac{T^d(-F)^{2-d/2}}{U_0 J^{d/2} L^{1-d/2}} \, .$$

cannot be meaningful.

At $d \geq 2$ the KPZ problem with $U(\mathbf{x}-\mathbf{x}') = U\delta(\mathbf{x}-\mathbf{x}')$ is ill defined and needs some regularization. One of the most natural regularizations consist in assuming that $U(\mathbf{x}-\mathbf{x}')$ has a finite correlation radius ξ.

FINITE-RANGE CORRELATIONS

The "wave-function" $\Psi(\mathbf{x})$ is localized at the bottom of the well ($E_0 \approx V(0) < 0$), whose form is given by

$$V(\mathbf{x}) = -\Lambda \int d\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\Psi^2(\mathbf{x}) \approx \frac{U(\mathbf{x})}{U(0)} V(0) \,.$$

This leads to

$$S \approx \frac{L}{2} \frac{[V(0)]^2}{U(0)} \approx \frac{(-F)^2}{2LU(0)}$$

Thus, for $\xi > 0$ the most distant part of the left tail is Gaussian and temperature-independent (for any d).

THE FAR-RIGHT TAIL

In contrast to the left tail, the blue terms in the integrand in

$$S = \frac{1}{2U_0} \int_0^t dt \int d\mathbf{x} \left[\frac{\partial f}{\partial t} + \frac{1}{2J} (\nabla f)^2 - \nu \nabla^2 f \right]^2$$

cannot cancel each other.

$$\partial f / \partial t \sim (\nabla f)^2 / J \quad \Rightarrow \quad \Delta(F) \sim (FL/J)^{1/2}.$$

$$S(F) \sim \frac{L\Delta^d}{U_0} \left(\frac{F}{L} \right)^2 \sim \frac{F^{2+d/2}}{U_0 J^{d/2} L^{1-d/2}} \ .$$

Since $\Delta(F)$ grows with F and L, the finiteness of ξ does not matter (as soon as $d < 2$).

At $d > 2$ the optimal fluctuation is localized both in transverse and longitudinal (along the string) directions and

$$S(F) \sim \frac{\xi^{d-2} F^3}{U_0 J} \ .$$

FIXED BOUNDARY CONDITIONS

$$\mathbf{x}(0) = \mathbf{x}(t) = 0$$

$$\lim_{t=0} \left[\mathbf{u}(t, \mathbf{x}) - \frac{\mathbf{x}}{t} \right] = 0$$

In both tails the scaling of F remains the same
(and in the left tail even the coefficient is not changed).

THE RENORMALIZATION EFFECTS

In terms of viscosity $\nu = T/2J$ and pumping force intensity $D = U_0/2J^2$ the answer for the far-left tail obtained for $d < 2$ and $U(\mathbf{x}) = U_0\delta(\mathbf{x})$ is

$$S(F) = \frac{\nu^d}{D} \frac{(-F/J)^{2-d/2}}{L^{1-d/2}} \, .$$

It is directly applicable only when $\Delta \ll x_0$, where

$$x_0^{2-d} \sim \frac{\nu^3}{D} \sim \frac{T^3}{JU_0} \, .$$

At larger Δ one should take into account the renormalization of ν and D (J is not renormalized due to the Galilean invariance).

THE STATIONARY STATE of the KPZ PROBLEM

In the strong-coupling phase

$$\langle [f(t,\mathbf{x}) - f(t',\mathbf{x}')]^2 \rangle \propto |\mathbf{x}-\mathbf{x}'|^{2\chi} g\left(\frac{|t-t'|}{|\mathbf{x}-\mathbf{x}'|^z}\right) .$$

χ is the roughening exponent of the KPZ problem and
z is the dynamic exponent.

$$\langle [\mathbf{x}(t) - \mathbf{x}(t')]^2 \rangle \propto (t-t')^{2/z} ,$$

$\zeta \equiv 1/z$ is the polymer's roughening exponent.

Sufficiently slow and not too strong fluctuations of $f(t,\mathbf{x})$
can be described in terms of effective viscosity: $\nu_{\text{eff}}(R) \sim \nu\,(R/a_\nu)^{2-z}$
and effective pumping force intensity: $D_{\text{eff}}(R) \propto R^{d+2\chi-z}$
(initially $D \equiv U_0/2J^2$).

Comparison of

$$D_{\mathsf{eff}}(R) \sim \int_{|r|<R} d\mathbf{r} \int_{-\infty}^{+\infty} d\tau \left[\overline{u^a(t,\mathbf{x})u^b(t+\tau,\mathbf{x}+\mathbf{r})}\right]^2$$

$$\sim \frac{R^{2+d}u^4_{\mathsf{typ}}(R)}{\nu_{\mathsf{eff}}(R)}\,.$$

with

$$u^2_{\mathsf{typ}}(R) \equiv \overline{\langle\mathbf{u}\rangle^2_R} \sim \frac{D_{\mathsf{eff}}(R)}{\nu_{\mathsf{eff}}(R)R^d}\,.$$

leads to

$$\frac{\nu^3_{\mathsf{eff}}(R)}{D_{\mathsf{eff}}(R)} \sim R^{2-d}\,,$$

which corresponds to

$$z + \chi = 2\,.$$

THE UNIVERSAL LEFT TAIL

In terms of $\nu = T/2J$ and $D = U_0/2J^2$ the answer for the far-left tail obtained at $d < 2$ and $U(\mathbf{x}) = U_0\delta(\mathbf{x})$ can be rewritten as

$$S(F) = \frac{\nu^d}{D}\frac{(-F/J)^{2-d/2}}{L^{1-d/2}}\,.$$

The replacement

$$\nu \to \nu_{\text{eff}}(\Delta)\,, \quad D \to D_{\text{eff}}(\Delta)$$

(where Δ has to be found self-consistently) then gives

$$S(F) \sim \left(\frac{-F}{F_*}\right)^{\eta_-}\,,$$

with

$$\eta_- = \frac{z}{2(z-1)}\,, \quad F_* \sim J\nu\left(\frac{\nu L}{a_\nu^2}\right)^{\omega} \text{ and } \omega = 1 - \frac{1}{\eta_-} = 2\zeta - 1\,,$$

$\zeta \equiv 1/z$ being the polymer's roughening exponent.

THE CONDITION FOR QUASI-STATIONARITY:

$$L \gg \tau(\Delta) \sim \Delta^2/\nu_{\mathrm{eff}}(\Delta)$$

is always fulfilled in the tail, because $S(F) \sim \frac{L}{\tau(\Delta)}$.

THE UNIVERSAL RIGHT TAIL

Here the renormalization should be stopped not at $R \sim \Delta(F)$, but at a smaller scale R_*, at which typical velocity of equilibrium fluctuations $u_{\mathrm{typ}}(R) \sim [D_{\mathrm{eff}}(R)/\nu_{\mathrm{eff}}(R)R^d]^{1/2}$ becomes comparable with the proper velocity of the optimal fluctuation $u_F \sim (F/JL)^{1/2}$.

$$R_*(F) \sim \Delta(-F) \ll \Delta(F) \; .$$

The replacement $D \to D_{\mathrm{eff}}[R_*(F)]$ then transforms

$$S(F) \sim \frac{(F/J)^{2+d/2}}{DL^{1-d/2}}$$

into

$$S(F) \sim \left(\frac{-F}{F_*}\right)^{\eta+} \; ,$$

with the same $F_*(L)$ as in the left tail and $\eta_+ = (1+d)\eta_-$.

THE SUMMARY

			η	ω
non-universal tails	far-left	$\xi = 0, \quad 0 < d < 2$	$\frac{4-d}{2}$	$\frac{2-d}{4-d}$
		$\xi > 0, \quad d > 0$	2	1/2
	far-right	$0 < d < 2$	$\frac{4+d}{2}$	$\frac{2-d}{4+d}$
		$d > 2, \quad \xi > 0$	3	0
universal tails	left		$\frac{1}{2(1-\zeta)}$	$2\zeta - 1$
	right		$\frac{1+d}{2(1-\zeta)}$	$2\zeta - 1$

$d = 1$

- Far-left tail: Zhang (1989) from replica approach.
- The full form of $P_L(F)$ in the universal regime: Prähofer and Spohn (2000) from the exact solution of the PNG model.