1d Anderson localization: devil's staircase of statistical anomalies

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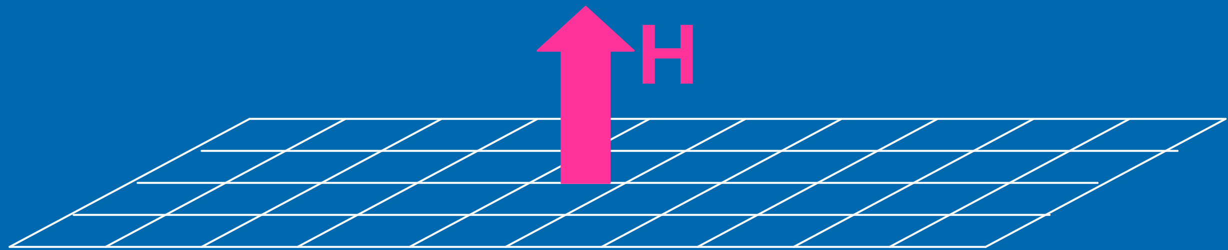
Hofstadter butterfly: hierarchy of spectral gaps



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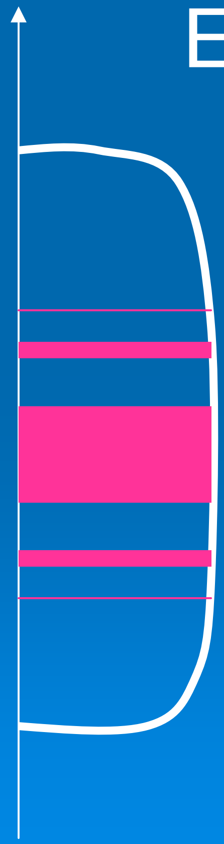
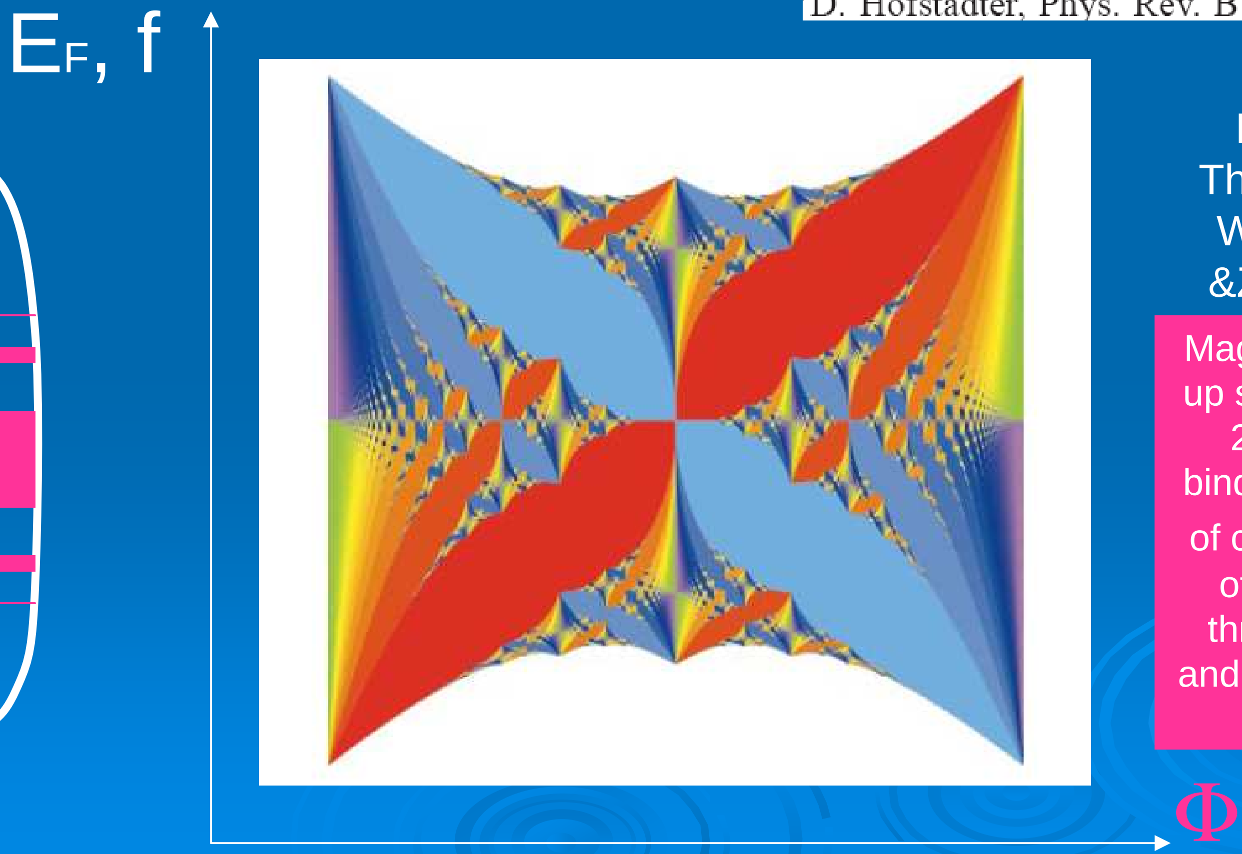
D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

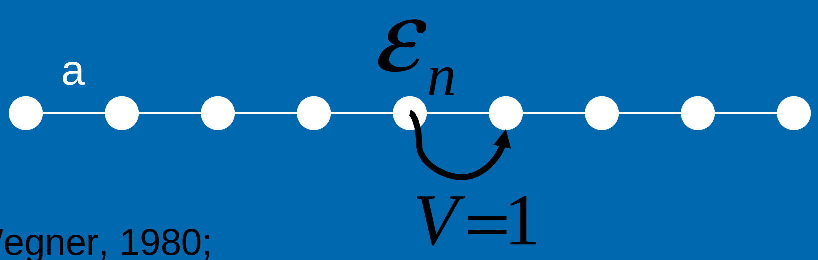


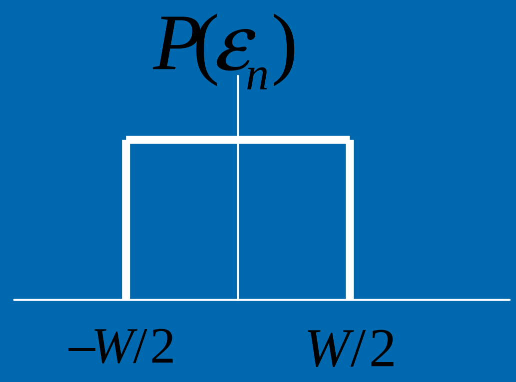


Harper,
Thouless…
Wigmann
&Zabrodin

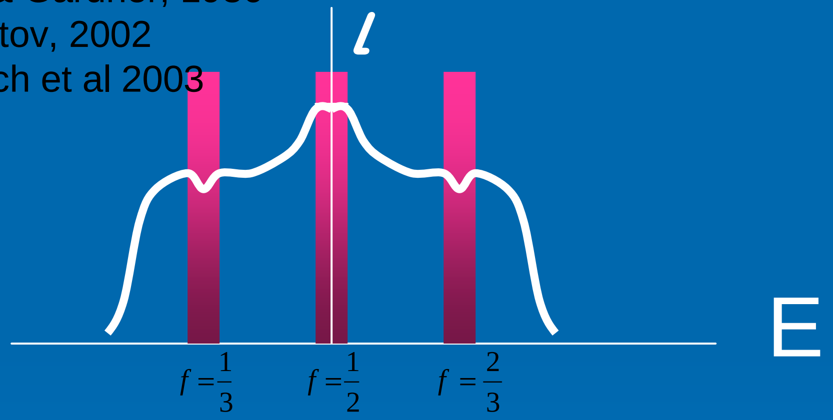
Magnetic field opens up spectral gaps in a 2D lattice tight-binding model: effect of commensurability of magnetic flux through a unit cell and the magnetic flux quantum

Anderson localization: hierarchy of statistical anomalies





F.Wegner, 1980;
Derrida & Gardner, 1986
Titov, 2002
Deytch et al 2003



Effect of commensurability of the lattice constant a and the Fermi-wavelength λ. This effect is not present in the continuous model.

Localization length sharply increases in the vicinity of E=0 and sharply decreases in the vicinity of E=1

At E=0 (f=1/2): $\delta\ell/\ell = 1.10$

The width of anomalous region $\sim W^2 << 1$

At E=1 (f=1/3 or 2/3): $\delta\ell/\ell \sim W^2 << 1$

What about the entire wavefunction statistics?

$$P(|\Psi|^2) \quad ?$$

Generalized Fokker-Planck equation

$$\left\langle |\Psi_n|^{2q} \right\rangle = \frac{2}{(q-2)!} \int_0^{\pi} \frac{d\phi}{\pi} \cos^{2q}\phi \int_0^{\infty} dz \;\; z^{q-2}\, \Phi_n(z,\phi)\, \Phi_{N-n}(z,-\phi-2\pi f)$$

$$\Psi \sim \sqrt{z}\cos(\phi)$$

Result of the super-symmetric quasi-sigma model (Ossipov, Kravtsov 2006)

The function Φ is found from the generalized Fokker-Planck equation:

$$\partial\Phi(u,\phi\,|\,x)/\partial x = [\hat{L}_f(u,\phi) - u] \;\; \Phi(u,\phi\,|\,x)$$

where $\Phi(u,\phi\,|\,x) = \Phi_{n=x\ell_0}(z\ell_0/2,\;\phi)$

$$\ell_0 = \frac{2\sin^2(\pi f)}{W^2}$$

Is the localization length without anomalous contributions

Where are the anomalous terms?

$$\partial\Phi(u,\phi\,|\,x)/\partial x = [\hat{L}_f(u,\phi) - u]\;\; \Phi(u,\phi\,|\,x)$$

$$\hat{L}_f(u,\phi) = \hat{L}_f^{(0)}(u,\phi) + W^2\hat{L}_f^{(1)}(u,\phi) + W^4\hat{L}_f^{(2)}(u,\phi) + \ldots$$

The term $\hat{L}_f^{(p)}(u,\phi)$ has a part that emerges only at f=m/(p+2)

$$\hat{L}_f^{(p)}(u,\phi) = \hat{L}_f^{(p,reg)}(u,\phi) + \sum_{m=1}^{p+1} \Delta\hat{L}_m^{(p)}(u,\phi)$$

Center of the band anomaly f=1/2

$$\partial_x \Phi(u,\phi\,|\,x) = \left[\hat{L}_f^{(0)}(u,\phi) - u\right] \Phi(u,\phi\,|\,x)$$

$$\hat{L}_f^{(0,reg)} = u^2 \partial_u^2 + \frac{3}{4}\partial_\phi^2$$

This term survives in the continuous limit

$$\Delta\hat{L}_f^{(0)} = -\cos(4\phi) u^2 \partial_u^2 + 2\cos(4\phi) u \partial_u - 2\cos(4\phi) + \\ + \sin(4\phi) u \partial_u \partial_\phi + \frac{1}{4}(3+\cos(4\phi))\partial_\phi^2 - \frac{3}{2}\sin(4\phi)\partial_\phi$$

This term is present only for f=1/2

$$\Psi \sim \sqrt{u}\cos(\phi)$$

Dependence of the Fokker-Plank equation on the phase at f=1/2

Is this a mess?

Exact integrability of the Fokker-Plank equation

N-> infinity: $$\Phi(u,\phi\,|\,x)=\Phi(u,\phi)$$

$$\left[\hat{L}_f^{(0)}(u,\phi)-u\right]\ \Phi(u,\phi)=0$$

obeys zero-mode Fokker-Planck Equation

$v=u\cos(2\phi)$ $\qquad$ $\Phi(u,\phi)=u\tilde{\Phi}(u,v)$

$$\sqrt{u^2-v^2}\ \left\{\ \partial_u\sqrt{u^2-v^2}\,\partial_u+\partial_v\sqrt{u^2-v^2}\,\partial_v\right\}\ \tilde{\Phi}(u,v)=\frac{u}{2}\tilde{\Phi}(u,v)$$

This equation is exactly integrable

Separation of variables and the inverse-square Hamiltonian

$$\Psi(u,v)=\sqrt[4]{u^2-v^2}\quad\tilde{\Phi}(u,v)\equiv\psi_\lambda(\xi)\psi_{-\lambda}(\eta)$$

$$\hat{H}(u,v)\Psi(u,v)=[\hat{h}(\xi)+\hat{h}(\eta)]\psi_\lambda(\xi)\psi_{-\lambda}(\eta)=0$$

$$\hat{h}(\xi)\psi_\lambda(\xi)\equiv\left[-\partial_\xi^2-\frac{3}{16\xi^2}+\frac{1}{4\xi}\right]\psi_\lambda(\xi)=\lambda\psi_\lambda(\xi)$$

$$\xi=u\cos^2(\phi)\qquad\eta=u\sin^2(\phi)$$

Celebrated inverse-square Hamiltonian

Problem of continuous degeneracy

$$\Phi(\xi,\eta) = \frac{\xi+\eta}{(\xi\eta)^{1/4}} \int_0^\infty d\lambda \, \frac{\tilde{C}(\lambda)}{\lambda^{\frac{3}{2}}} \left[W_{-\lambda\epsilon,\frac{1}{4}} \left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) W_{-\lambda\bar{\epsilon},\frac{1}{4}} \left(\frac{\epsilon\eta}{4\lambda}\right) + c.c \right]$$

where $\epsilon = e^{i\pi/4}$, $\bar{\epsilon} = e^{-i\pi/4}$, and $\tilde{C}(\lambda)$ is a real function such that $\tilde{C}(\lambda \to 0) = \text{const}$

W is the Whittaker function

$\xi = u\cos^2(\phi)$ $\eta = u\sin^2(\phi)$

Q: **How to fix the function C(λ)?**

A: Smoothness of Φ(u,ϕ) with all the derivatives

+ some miraculous properties of the Whittaker function

Final result for the generating function

$$\Phi(\xi,\eta) = \frac{\xi+\eta}{(\xi\eta)^{1/4}} \int_0^\infty d\lambda \, \frac{\tilde{C}(\lambda)}{\lambda^{\frac{3}{2}}} \left[W_{-\lambda\epsilon,\frac{1}{4}} \left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) W_{-\lambda\bar{\epsilon},\frac{1}{4}} \left(\frac{\epsilon\eta}{4\lambda}\right) + c.c \right]$$

$$\xi = u\cos^2(\phi) \qquad \tilde{C}(\lambda) = A\,\Gamma\left(\frac{1}{4}+\epsilon\lambda\right)\Gamma\left(\frac{1}{4}+\bar{\epsilon}\lambda\right) \qquad \eta = u\sin^2(\phi)$$

WHY SO MUCH OF A MIRACLE?

WHAT IS THE HIDDEN SYMMETRY?

What is the new properties of the anomalous statistics?

Conclusion

- Statistical anomalies in 1d Anderson model at any rational filling factor f.
- Integrability of the TM equation for GF determining all local statistics at a principal anomaly at f=1/2.
- Unique solution for the GF in the infinite 1d Anderson model.
- Hidden symmetry that makes TM equation integrable?

$$\Delta\hat{L}^{(1)} = \delta_{m,3}\left[\frac{\sqrt{3}}{8}\cos(6\phi)\left(12x^2\partial_x^2\partial_\phi - \partial_\phi^3 - 36x\partial_x\partial_\phi + 44\partial_\phi\right) + \right.$$

$$\left. + \frac{\sqrt{3}}{4}\sin(6\phi)\left(4x^3\partial_x^3 - 3x\partial_x\partial_\phi^2 - 12x^2\partial_x^2 + 6\partial_\phi^2 + 24x\partial_x - 24\right)\right].$$