Anderson localization at boundaries: Classification of topological insulators and superconductors

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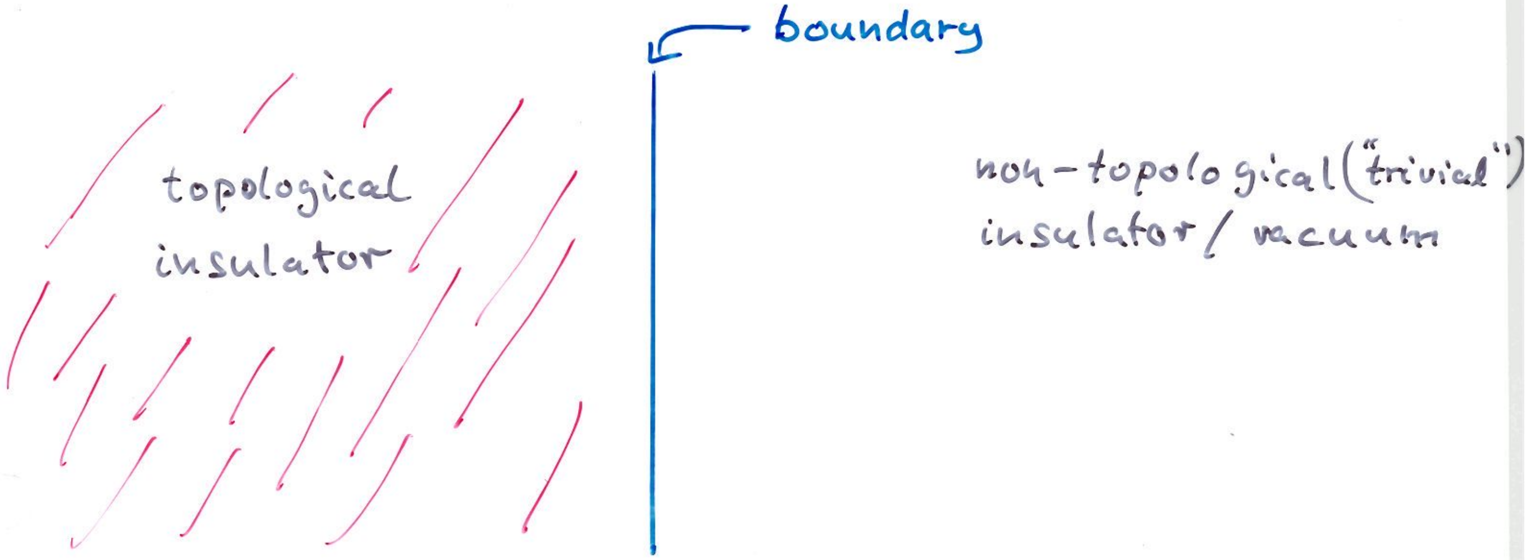
THEME OF THIS TALK:

Classifying topological properties of bulk insulators (or superconductors) by looking at their boundaries



A signature of the topological properties of the bulk is the appearance of gapless boundary degrees of freedom, which are entirely robust to perturbations [respecting the symmetries of the system (← more specific shortly)], including disorder.

THIS TALK:

We will use the appearance of robust gapless degrees of freedom on the sample boundaries, which cannot be Anderson localized by disorder, as a diagnostic of the topological properties of the bulk insulator.

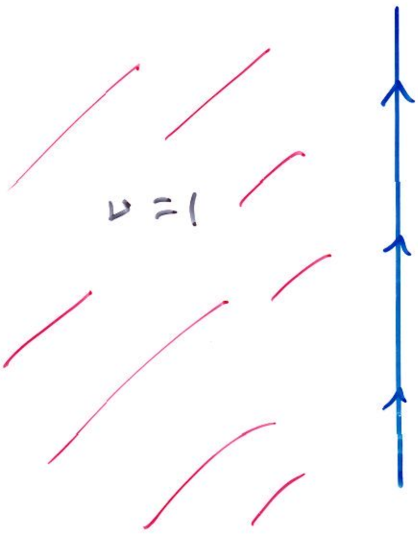
This talk:

No interactions

(but due to gap: stable to sufficiently weak interactions)

SOME EXAMPLES:

$d=2$: Integer Quantum Hall insulator ($\mathcal{T}$ broken)



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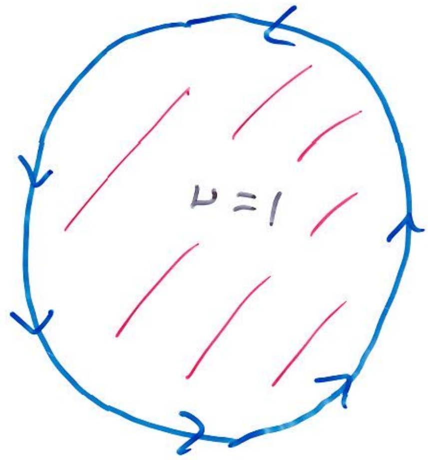
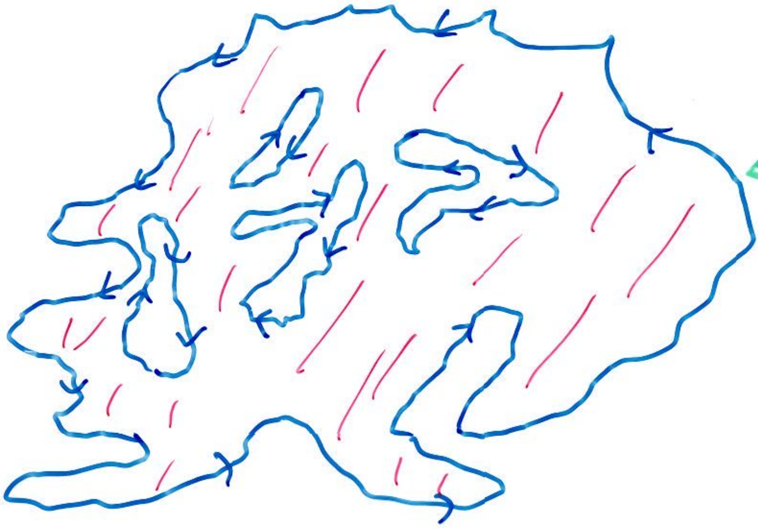
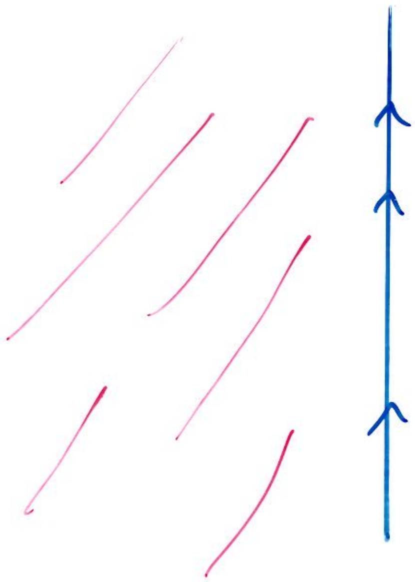


Here: because the direction of propagation at the edge is in one direction (due to chirality from $\mathcal{T}$-breaking) the edge state cannot (trivially) be localized by disorder.



$\leftarrow$ {a disordered Quantum Hall insulator}

$d = 2$: Chiral $(p_x + i p_y)$ superconductor (T broken)



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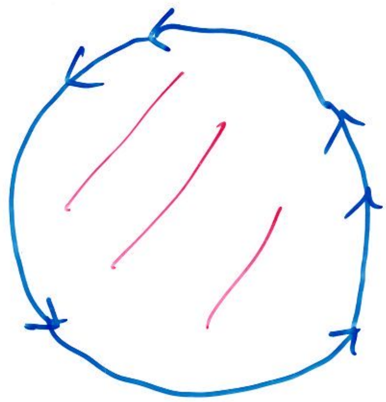
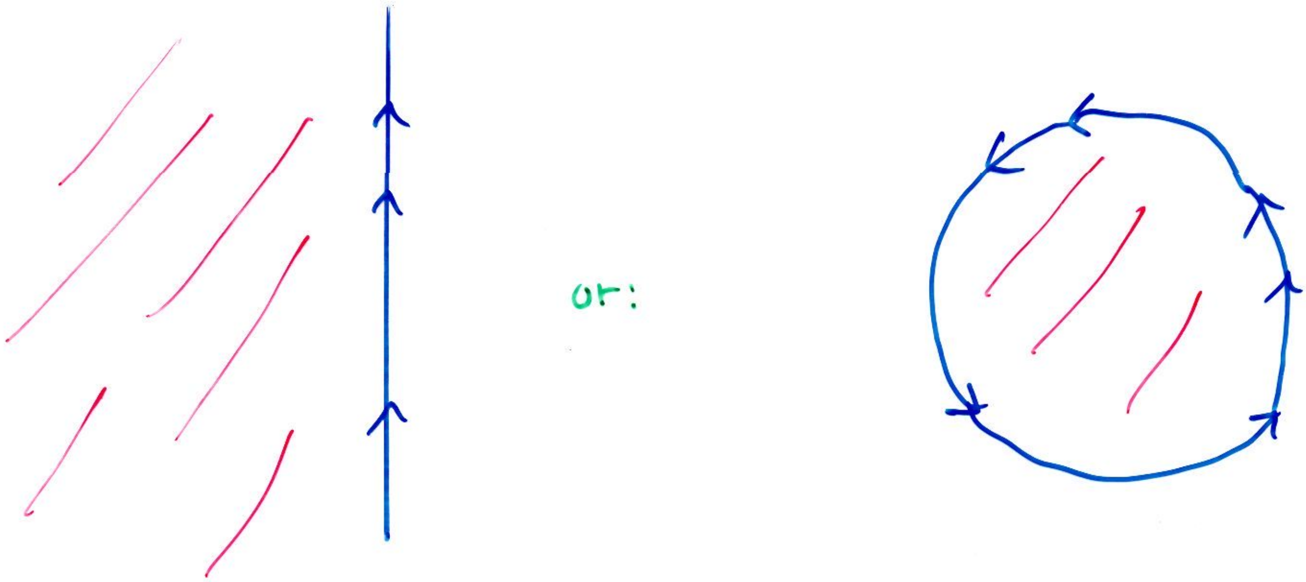


Similar to Quantum Hall insulator, but no charge transport occurs at the edge, only heat transport [different "symmetries" → more specific shortly].

$d=2$: Chiral $(p_x + i p_y)$ superconductor (T broken)



Similar to Quantum Hall insulator, but no charge transport occurs at the edge, only heat transport [different "symmetries" → more specific shortly].

→: experiments: HgTe, BiSb materials

$d=2$: $\mathbb{Z}_2$ - topological insulator

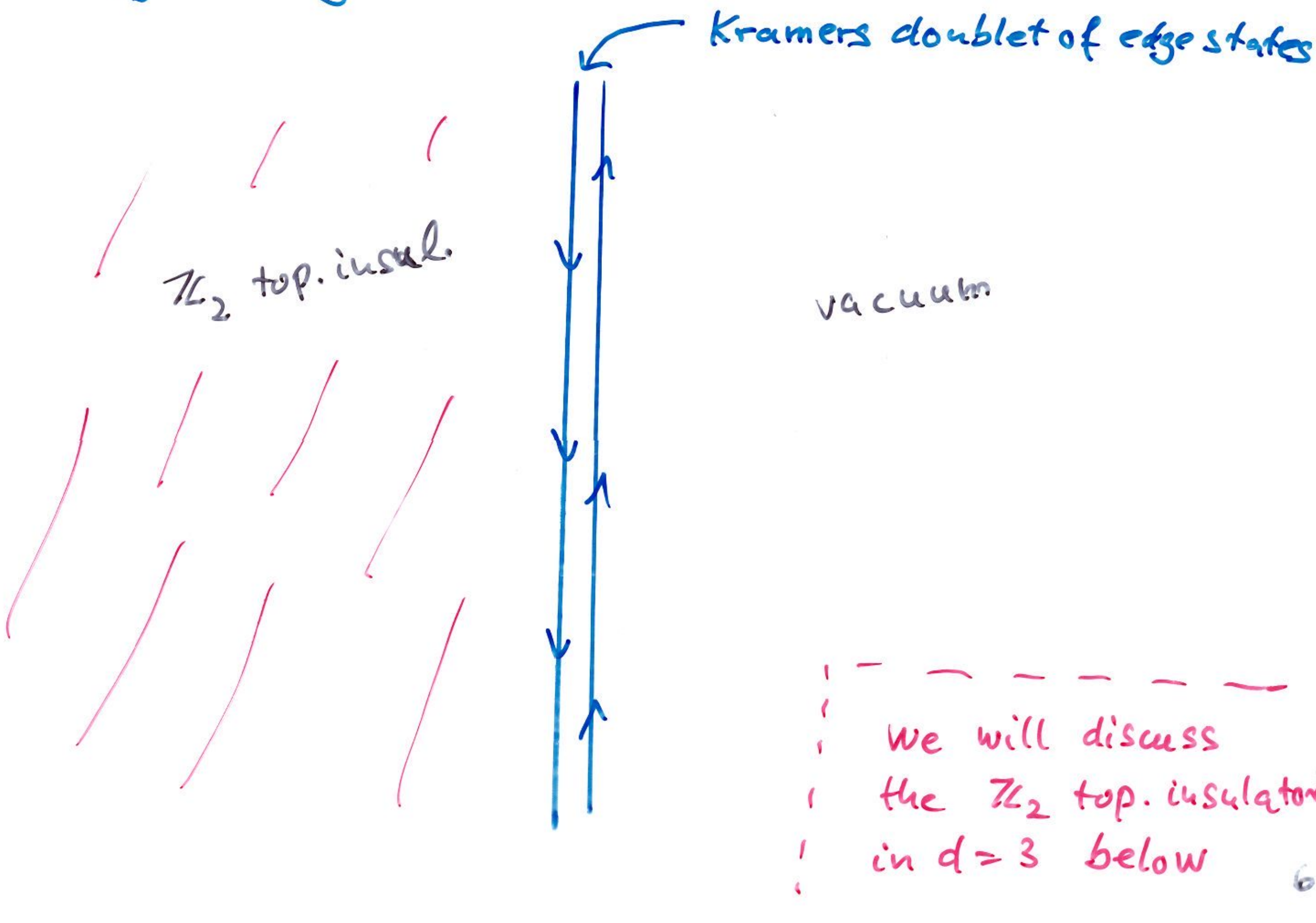
More recently it was realized that gapped phases supporting topologically protected gapless states appearing at the sample boundaries may also appear in the absence of time reversal symmetry breaking.

These occur in certain band insulators with strong spin-orbit interactions, and are known as $\mathbb{Z}_2$ topological insulators, the "quantum spin Hall state" (QSH).

[work by Kane, Mele and many others].

→: A certain non-trivial topological invariant can be associated with these bulk states.

→: In $d=2$, say, $\mathbb{Z}_2$ topological insulators are known to possess gapless edge states consisting of a Kramers doublet.
Even though there is a forward and a backward propagating mode, these cannot be Anderson localized even by strong disorder.



We will discuss the $\mathbb{Z}_2$ top. insulator in $d=3$ below

In this talk we ask the general questions:

→: Which systems possess gapped ground states with topologically non-trivial properties?

→: How many such systems are there?

→: Specifically, we may consider continuously deforming the ground state, giving rise to a gapped phase.
We then ask, how many different such phases a system can possess, so that in going from one phase to another a quantum phase transition has to be crossed.

→: Because of the presence of the bulk gap, the bulk phases are robust against disorder.

Thus one is led to seek a classification of ground states of (in general) random gapped Hamiltonians.

There are only 10 classes of random Hamiltonians.

This underlies the well known classification of random matrices, and universality classes of Anderson localization transitions.

[Zirnbauer (1996), Altland + Zirnbauer (1997), Heinzner, Huckleberry, Zirnbauer (2004)]

Brief Review: Classification of random Hamiltonians – the "10-fold way"

In classifying random Hamiltonians one must consider only the most generic symmetries,

time reversal (T)

and

charge-conjugation (particle-hole) (C)

There are only 10 possible behaviors of a Hamiltonian under T and C.
(10 "symmetry classes")

The basic idea is simple to understand:

→: T is antiunitary: $T = \mathcal{U}_T \cdot K$

unitary

complex conjugation

$\mathcal{H}$ = 1st quantized Hamiltonian

$T: \quad \mathcal{U}_T \, \mathcal{H}^* \, \mathcal{U}_T^{\dagger} = \mathcal{H}$

[SU(2) angular momentum = integer]

$$T = \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and } T^2 = +\mathbb{1} \\ -1 & \text{time reversal invariance and } T^2 = -\mathbb{1} \end{cases}$$

[SU(2) angular momentum = $\frac{1}{2}$-integer]

→: C is antiunitary: $C = \mathcal{U}_C \cdot K$

$C: \quad - \mathcal{U}_C \, \mathcal{H}^* \, \mathcal{U}_C^{\dagger} = \mathcal{H}$

$$C = \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and } C^2 = +\mathbb{1} \\ -1 & \text{particle-hole symmetry and } C^2 = -\mathbb{1} \end{cases}$$

Note:

* There are $3 \times 3 = 9$ choices for $T \times C$

* For 8 of these choices the value of $S = TC$ is uniquely fixed: these are all except for "A" and "A$\underline{\text{III}}$".

* For "A" and "A$\underline{\text{III}}$":

T	C
0	0

$\Rightarrow$ free to choose $S = 0$ or 1, yielding "A" and "A$\underline{\text{III}}$".

TABLE – "10 fold way"

(CARTAN) Name	T	C	S = CT	Hamiltonian $\mathcal{H}$ element of	NLσM Manifold (Fermionic Replicas)
A (unitary)	0	0	0	$U(N)$	$\frac{U(2N)}{U(N)\times U(N)}$
AI (orthogonal)	+1	0	0	$U(N)/O(N)$	$\frac{Sp(4N)}{Sp(2N)\times Sp(2N)}$
AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$	$\frac{O(2N)}{O(N)\times O(N)}$
AIII (chiral unitary)	0	0	1	$\frac{U(N+M)}{U(N)\times U(M)}$	$U(N)$
BDI (chiral orthogonal)	+1	+1	1	$\frac{O(N+M)}{O(N)\times O(M)}$	$\frac{U(2N)}{Sp(2N)}$
CII (chiral symplectic)	−1	−1	1	$\frac{Sp(2N+2M)}{Sp(2N)\times Sp(2M)}$	$\frac{U(N)}{O(N)}$
D	0	+1	0	$O(N)$	$\frac{O(2N)}{U(N)}$
C	0	−1	0	$Sp(2N)$	$\frac{Sp(2N)}{U(N)}$
DIII	−1	+1	1	$\frac{SO(2N)}{U(N)}$	$O(N)$
CI	+1	−1	1	$\frac{Sp(2N)}{U(N)}$	$Sp(2N)$

TABLE – "10 fold way" ['CARTAN Classes']

(CARTAN) Name	T	C	$S = CT$	Hamiltonian $\mathcal{H}$ element of	NLσM Manifold (Fermionic Replicas)	SU(2) SPIN CONSERVED	Examples
A (unitary)	0	0	0	$U(N)$	$U(2N)/U(N)\times U(N)$	YES/NO	•IQHE •Anderson
AI (orthogonal)	+1	0	0	$U(N)/O(N)$	$Sp(4N)/Sp(2N)\times Sp(2N)$	YES	•Anderson
AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$	$O(2N)/O(N)\times O(N)$	NO	•Quantum Spin Hall: $\mathbb{Z}_2$-Top. Insulator •Anderson (spin-orbit)
AIII (chiral unitary)	0	0	1	$U(N+M)/U(N)\times U(M)$	$U(N)$	YES/NO	•Random Flux •Gade
BDI (chiral orthogonal)	+1	+1	1	$O(N+M)/O(N)\times O(M)$	$U(2N)/Sp(2N)$	YES	•Bipartite Hopping •Gade •Hatsugai-Wen-Kohmoto
CII (chiral symplectic)	−1	−1	1	$Sp(2N+2M)/Sp(2N)\times Sp(2M)$	$U(N)/O(N)$	NO	•Bipartite Hopping •Gade
D	0	+1	0	$O(N)$	$O(2N)/U(N)$	NO	•(p_x+ip_y)-wave 2D •S.C. w/ spin-orbit •TQHE
C	0	−1	0	$Sp(2N)$	$Sp(2N)/U(N)$	YES	•singlet SC $(d+id)$-wave •SQHE
DIII	−1	+1	1	$SO(2N)/U(N)$	$O(N)$	NO	•He 3B •S.C. with Spin-orbit
CI	+1	−1	1	$Sp(2N)/U(N)$	$Sp(2N)$	YES	•singlet SC

COMMENT:

SUPERCONDUCTORS:

$\mathcal{H}$ = Bogoliubov - De Gennes Hamiltonian
for quasiparticles
(within MFT of pairing)

↑

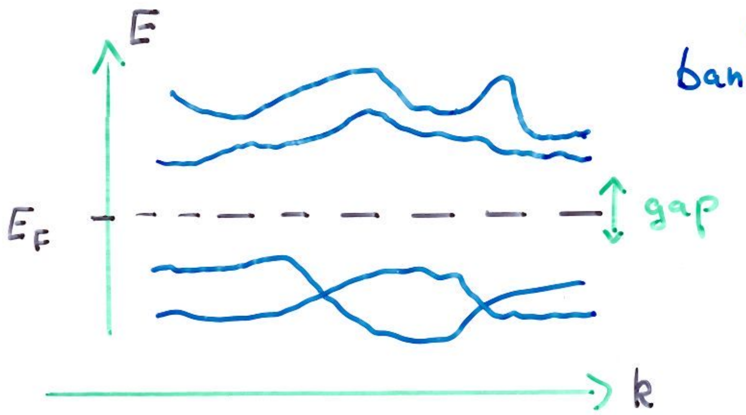
(has natural particle-hole symmetry)

Explicit construction of topological invariants — translationally invariant case

→: Translational invariance ⇒
ground states of non-interacting Fermions (= band insulators)
are filled Fermi seas in the Brillouin zone:

k = momentum : $\boxed{\mathcal{H}(k)\,|u_a(k)\rangle = E_a(k)\,|u_a(k)\rangle}$

band index



→: Consider Projector onto filled Bloch states:

$$\boxed{P(k) := \sum_a^{\text{filled}} |u_a(k)\rangle\,\langle u_a(k)|}$$

SPECTRAL PROJECTOR

→: Convenient to define: $\boxed{Q(k) := 2P(k) - \mathbb{1}}$

properties: $Q^\dagger = Q$, $Q^2 = \mathbb{1}$, $\mathrm{tr}\, Q = m - n$

filled ↗ ↖ empty

(Q = Hamiltonian where $E_a(k)$ ↗ −1 empty ↘ +1 full)

→: Consider the case of a Hamiltonian without any symmetry conditions (for simplicity): class A

class A: the Hamiltonian $\mathcal{H}$ is a general Hermitean matrix

-: set of eigenvectors = arbitrary unitary matrix ∈

$$\in \mathcal{U}(m+n)$$

filled empty

-: gauge symmetry: $\mathcal{U}(m) \times \mathcal{U}(n)$ = relabeling filled and empty states

-: ⇒ $Q(k) \in \mathcal{U}(m+n) / \mathcal{U}(m) \times \mathcal{U}(n)$ = "Grassmanian"

$$Q: \quad BZ \longrightarrow \mathcal{U}(m+n) / \mathcal{U}(m) \times \mathcal{U}(n)$$
$$k \longrightarrow Q(k)$$

The Quantum ground state is a map from the Brillouin Zone into the Grassmanian

$\Rightarrow$: How many inequivalent (= not deformable into each other) groundstates (= maps) are there?

This is answered by the Homotopy Group:

—: <u>In $d=2$</u>: $\pi_2\left[U(m+n)/U(m)\times U(n)\right] = \mathbb{Z}$ =

= counts the number of edge states of $d=2$ integer Quantum Hall states

—: <u>In $d=3$</u>: $\pi_3\left[U(m+n)/U(m)\times U(n)\right] = \{1\}$ (for sufficiently large m, n)

$\Rightarrow$: $\boxed{\text{There are } \underline{\underline{\text{no}}} \text{ topological insulators in } d=3 \text{ dimensions in symmetry class A}}$

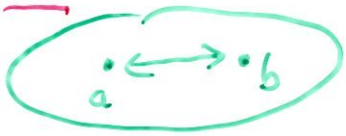
→: The spectral projector has the same symmetries as the Hamiltonian.

E.g.: The T-symmetry in the spin-orbit symmetry class AII implies the constraint:

$$\sigma_y \, Q(-k)^* \, \sigma_y = Q(k)$$

Even though there are no different topological sectors for $Q(k)$ in the absence of the constraint, there may be some in the presence of the constraint:

A: no constraint



a, b can be deformed into each other in the absence of the constraint

AII: constraint on $Q(k)$

a, b can no longer be deformed into each other in the presence of the constraint

⇑

indeed: AII is $\mathbb{Z}_2$ top. ins. ⟷

⟷ two inequivalent sectors

The sublattice symmetry S is a source of non-trivial topological sectors:

TABLE: $\Rightarrow$ There are 5 symmetry classes possessing sublattice symmetry S:

S = unitary: $\quad S \mathcal{H} S^\dagger = -\mathcal{H}$

$$S^2 = \mathbb{1}$$

$\Rightarrow$ can show: projector $Q(k)$ has block off-diagonal form in some basis

$$Q(k) = \begin{bmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{bmatrix}, \qquad q(k)\, q(k)^\dagger = \underline{\mathbb{1}}$$

(from $Q(k)^2 = \underline{\mathbb{1}}$)

$\rightarrow$:

symmetry classes with SLS	S	constraint
$A\underline{\overline{III}}$	1	none
$BD\underline{\overline{I}}$	1	$q^*(-k) = q(k)$
$C\underline{\overline{II}}$	1	$\sigma_y\, q(-k)^*\sigma_y = -q(k)$
$D\underline{\overline{III}}$	1	$q(-k)^t = -q(k)$
$C\underline{\overline{I}}$	1	$q(-k)^t = q(k)$

$\rightarrow$: Consider $A\underline{\overline{III}}$ (no constraint) : $q(k) \in \mathcal{U}(m)$
unconstraint

-: $$q : \begin{array}{ccc} BZ & \longrightarrow & \mathcal{U}(m) \\ k & \longrightarrow & q(k) \end{array}$$

-: $\pi_3\left[\mathcal{U}(m)\right] = \mathbb{Z}$ $\Rightarrow$ $d=3$ topological insulators in symmetry class $A\underline{\overline{III}}$ labeled by integer $\nu(q) \in \mathbb{Z}$

-: Explicit form:

$$\nu(q) = \frac{1}{24\pi^2} \int_{BZ} d^3k\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]$$

$\rightarrow$: In remaining four symmetry classes with SLS (i.e.: BDI, CII, DIII, CI) certain integers $\nu(q)$ may not be allowed, due to the constraints.

We will find the answer to this below by counting the number of robust gapless modes appearing at the 2D boundary

Result:

AIII and DIII : $\nu \in \mathbb{Z}$ (no change)

CI : $\nu \in 2\mathbb{Z}$

CII and BDI : $\nu = 0$ $\left\{ \begin{array}{l} \text{actually:} \\ \text{CII is } \mathbb{Z}_2 \text{ top.} \\ \text{insulator} \\ \quad \rightarrow \text{below} \end{array} \right\}$

SUMMARY: Topological Insulators – so far

(Kane-Mele quantum spin Hall state)

	T	C	S		d=2	d=3
A	0	0	0		$\mathbb{Z}$	–
AI	+1	0	0			
AII	−1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII	0	0	1			$\mathbb{Z}$
BDI	+1	+1	1			
CII	−1	−1	1			$\mathbb{Z}_2$
D	0	+1	0		$\mathbb{Z}$	
C	0	−1	0		$\mathbb{Z}$	
DIII	−1	+1	1		$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1			$\mathbb{Z}$

Gapless degrees of freedom at $d=2$ boundary terminating the $d=3$ bulk insulator

→: Here: want to investigate robustness of of gaplessness against

(i) spatially homogeneous

(ii) random

perturbations of the Hamiltonian respecting the symmetries of the given class.

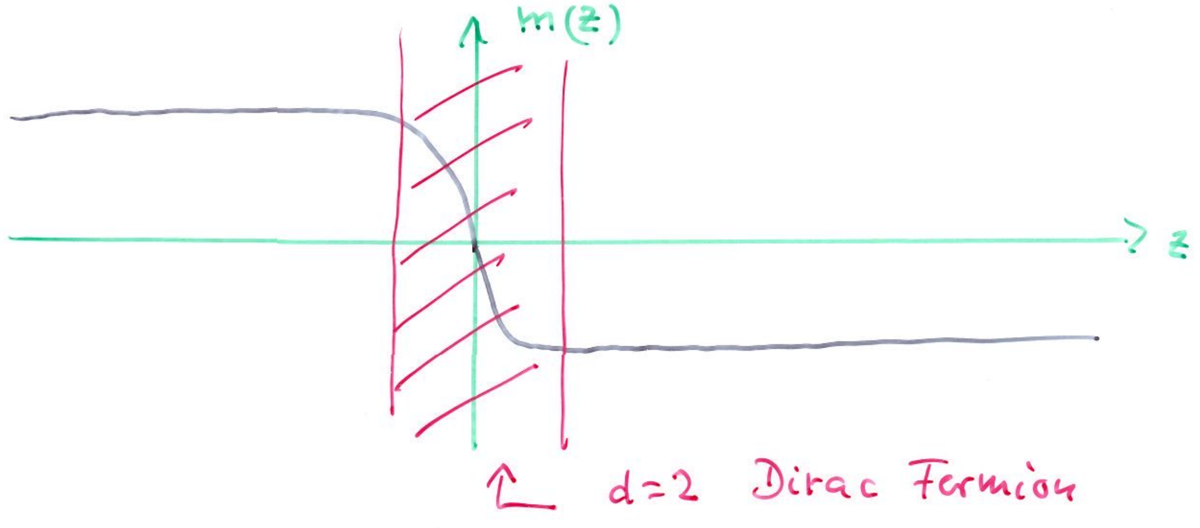
→: Need to look at 10 symmetry classes at $d=2$ boundary

→: Need one extra ingredient:

→: Can construct $d=3$ Dirac Hamiltonian representatives of $d=3$ top. insulators.

→: It is known* that changing sign of Dirac mass in one spatial direction introduces $d=2$ Dirac Fermions at the domain wall: ⇝

(*: Callan + Harvey (1985), Haldane (1988), Ludwig + Fisher + Shankar + Grinstein (1994), ...)



—: Therefore should allow for d=2 Dirac Fermions at the boundary (this is important).

RESULT: CLASSIFICATION OF d=2 DIRAC HAMILTONIANS Bernard + Le Clair (2001) [BL]:

$$\mathcal{H} = \left[\begin{array}{c|c} V_+ + V_- & -i\partial_{\bar{z}} + A_+ \\ \hline +i\partial_z + A_- & V_+ - V_- \end{array}\right] , \quad \left(\begin{array}{c} \text{matrices:} \\ V_\pm = V_\pm^\dagger \\ A_+^\dagger = A_- \end{array}\right)$$

* There exist 13 symmetry classes, not only 10
* Specifically: In each of AIII, DIII, CI there is an extra, new symmetry class, allowed by the Dirac structure (1st derivative Hamiltonian)

→: Dirac description of $d=2$ boundary degrees of freedom is very convenient and important in connection with topological properties of $d=3$ bulk insulator:

- Useful to review the $d=3$ $\mathbb{Z}_2$ top. insulator case (spin-orbit / symplectic symmetry class $A\mathrm{II}$):

* BL tells us that there exists a $d=2$ Dirac Hamiltonian in $A\mathrm{II}$ with $N_f=1$ flavor of gapless (2-component) Dirac Fermions.

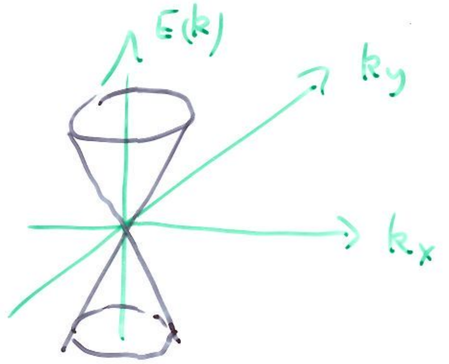
* A single flavor ($N_f=1$) cannot be realized on a $d=2$ lattice.
Thus, this situation must correspond to the boundary of a $d=3$ top. insulator in $A\mathrm{II}$.

* This single flavor ($N_f=1$) Dirac Fermion was constructed explicitly by Fu-Kane-Mele.

* BL show that the most general $N_f = 1$ flavor $d = 2$ Hamiltonian in AII is :

$$\boxed{\mathcal{H} = 2\times 2 \text{ matrix} = -i\left[\sigma_x \partial_x + \sigma_y \partial_y\right] + V(x,y)\cdot \mathbb{1}_{2\times 2}}$$

(single flavor ($N_f = 1$) 2-component Dirac Fermion)



⇐ a single Dirac cone

(• this is in class AII (spin-orbit) [Ludwig et al. (1994), Ando et al. (1998), ...])

• $V(x,y)$ = homogeneous or random cannot localize the Fermion

[Ryu* et al. (2007), Nomura et al (2007) ; Ostrovsky et al. (2007), Bardason et al. 2007]

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* $\mathbb{Z}_2$ - top. term in $\frac{O(2N)}{O(N)\times O(N)}$ NLσM prevents appearance of localized phase

Proof of gaplessness of d=2 boundary degrees of freedom in classes AIII, DIII, CI

* Precisely for $\boxed{AIII,\ DIII,\ CI}$ there are extra symmetry classes for d=2 Dirac (BL - as mentioned):

$(n = 1, 2, 3, \ldots)$

CARTAN CLASS	T	C	S	Bernard/ /LeClair (BL)	N_f = # of Fermion flavors
AIII	0	0	1	$(AIII)_a$	$(2n-1)$
				$(AIII)_b$	$2n$
DIII	-1	+1	1	$(DIII)_a$	$(2n-1)$
				$(DIII)_b$	$2n$
CI	+1	-1	1	$(CI)_a$	$(2n-1)\cdot 2$
				$(CI)_b$	$2n \cdot 2$

* In the extra classes $(AIII)_a$, $(DIII)_a$, $(CI)_a$ the d=2 Hamiltonian must be of the form

$$\boxed{\mathcal{H} = \left[\begin{array}{c|c} 0 & -i\partial_{\bar z} + A_+ \\ \hline -i\partial_z + A_- & 0 \end{array}\right]}$$

where: $\dim A_\pm = \begin{cases} (2n-1) \\ (2n-1)\cdot 2 \end{cases}$ for $\begin{Bmatrix} (AIII)_a,\ (DIII)_a \\ (CI)_a \end{Bmatrix}$

and $A_{\pm}$ = gauge potentials in the classical groups :

$$\left\{ \begin{array}{lcl} \mathcal{U}(2n-1) & & (A\,\mathrm{III})_a \\ SO(2n-1) & \text{for} & (D\,\mathrm{III})_a \\ Sp[2(2n-1)] & & (C\,\mathrm{I})_a \end{array} \right\}$$

* The gauge potentials, whether homogeneous or random, cannot localize the gapless Dirac Fermions

[Ludwig et al. (1994), Mudry et al. (1996), Tsvelik (1997), Ludwig (2000)]

* The number of robust gapless 2-component Dirac Fermion flavors N_f is a topological invariant of the $d=3$ gapped bulk insulator.

* Experimental signature :

|| universal longitudinal surface conductivity ||

→: On the other hand, it is very easy to show from the BL classification that in all other classes (except $\boxed{A\,\mathrm{II}}$ discussed above, and $C\,\mathrm{II}$ discussed below), the corresponding $d=2$ Dirac Hamiltonian can be made fully gapped while remaining in the respective symmetry class.

→: Class $\boxed{C\,\mathrm{II}}$: One can show, using BL, that class $C\,\mathrm{II}$ cannot localize iff

$N_f = 2 \bmod (4)$ [twice an odd integer]

UPDATED TABLE: Topological Insulators

	T	C	S		d=2	d=3	
A	0	0	0		$\mathbb{Z}$	–	
AI	+1	0	0		–	–	
AII	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	
AIII	0	0	1		–	$\mathbb{Z}$	
BDI	+1	+1	1		–	–	
CII	-1	-1	1		–	$\mathbb{Z}_2$	
D	0	+1	0		$\mathbb{Z}$	–	
C	0	-1	0		$\mathbb{Z}$	–	
DIII	-1	+1	1		$\mathbb{Z}_2$	$\mathbb{Z}$	
CI	+1	-1	1		–	$\mathbb{Z}$	

Classification of topological insulators in $d=1$

Same basic idea as above:

A diagnostic of a $d=1$ topological insulator is the appearance of gapless degrees of freedom at its boundaries.

In $d=1$ boundaries are points.

Question: In which symmetry classes are there gapless modes
(= zero modes = states at zero energy)
at a point

Dmitri Ivanov (zero modes in random matrix theory)
(2001)

Results:

$$D\,\mathrm{III}\,[\text{-odd}]: \quad \mathcal{H} \in SO(4N+2)/U(2N+1)$$

$$D\,[\text{-odd ("B")}]: \quad \mathcal{H} \in SO(2N+1)$$

"Majorana" zero mode

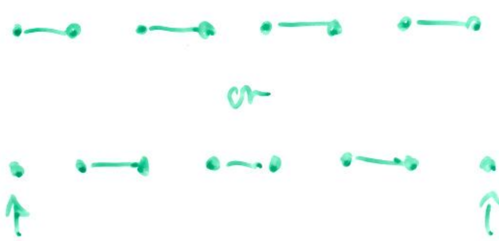
$$A\,\mathrm{III}: \quad \mathcal{H} \in U(p+q)/U(p)\times U(q)$$

$$BDI: \quad \mathcal{H} \in SO(p+q)/SO(p)\times SO(q)$$

$$C\,\mathrm{II}: \quad \mathcal{H} \in \frac{Sp[2(p+q)]}{Sp(2p)\times Sp(2q)}$$

$(q-p)$ zero modes – Verbaarschot (1994)

sublattice symmetry: appear for one dimerization:



RESULT:

FULLY UPDATED TABLE: Topological Insulators in $d = 1, 2, 3$

	T	C	S		d=1	d=2	d=3
A	0	0	0		–	$\mathbb{Z}$	–
AI	+1	0	0		–	–	–
AII	−1	0	0		–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII	0	0	1		$\mathbb{Z}$	–	$\mathbb{Z}$
BDI	+1	+1	1		$\mathbb{Z}$	–	–
CII	−1	−1	1		$\mathbb{Z}$	–	$\mathbb{Z}_2$
D	0	+1	0		$\mathbb{Z}_2$	$\mathbb{Z}$	–
C	0	−1	0		–	$\mathbb{Z}$	–
DIII	−1	+1	1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1		–	–	$\mathbb{Z}$

{LOOK AT SIMPLY REORDERED TABLE}

"SHIFT"

see very recently all dimensions "d": Alexei Kitaev's talk

	d=1	d=2	d=3
A	–	$\mathbb{Z}$	–
AIII	$\mathbb{Z}$	–	$\mathbb{Z}$
AI	–	–	–
BDI	$\mathbb{Z}$	–	–
D	$\mathbb{Z}_2$	$\mathbb{Z}$	–
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	$\mathbb{Z}$	–	$\mathbb{Z}_2$
C	–	$\mathbb{Z}$	–
CI	–	–	$\mathbb{Z}$