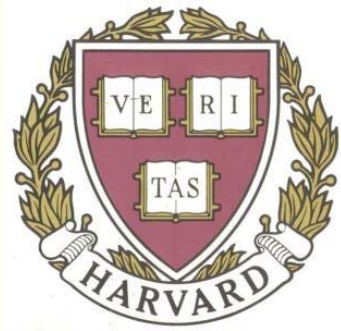


# Relativistic magnetotransport in graphene



Markus Müller

in collaboration with

Lars Fritz (Harvard)

Subir Sachdev (Harvard)

Jörg Schmalian (Iowa)



Landau Memorial Conference June 26, 2008



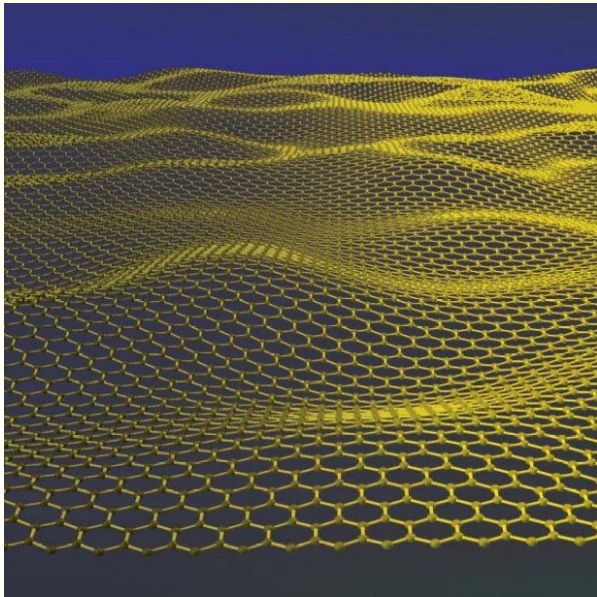
# Outline

- Relativistic physics in graphene, quantum critical systems and conformal field theories
  - Relativistic signatures in magnetotransport: el.+th. conductivity, Peltier, Nernst effect etc.
- Hydrodynamic description
  - Collective, collision-broadened cyclotron resonance
- Boltzmann equation
  - Recover and refine hydrodynamics with Boltzmann
  - Describe relativistic-to-Fermi liquid crossover
  - Go beyond hydrodynamics

# Dirac fermions in graphene

*(Semenoff '84, Haldane '88)*

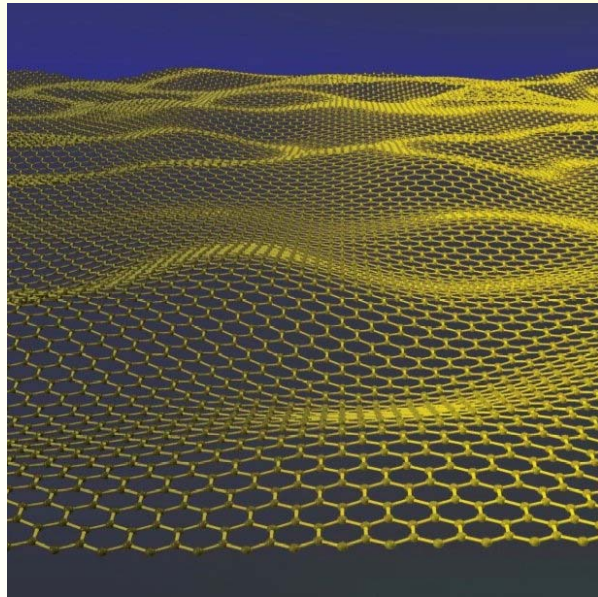
Honeycomb lattice of C atoms



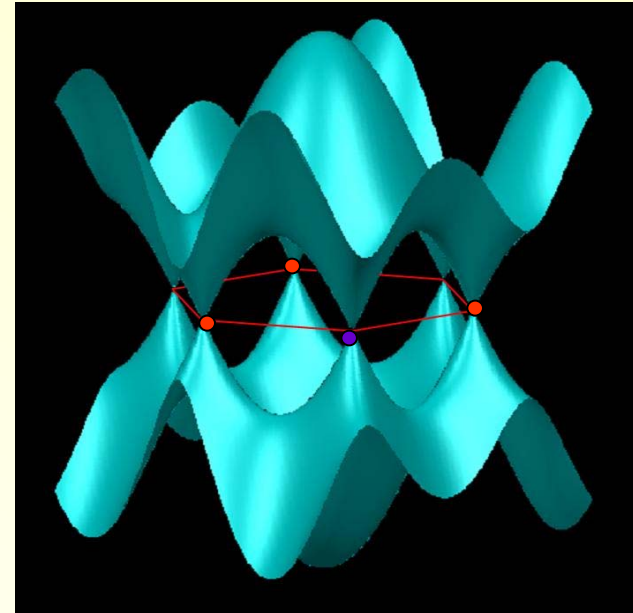
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Tight binding dispersion



2 massless Dirac cones in  
the Brillouin zone:  
(Sublattice degree of  
freedom  $\leftrightarrow$  pseudospin)

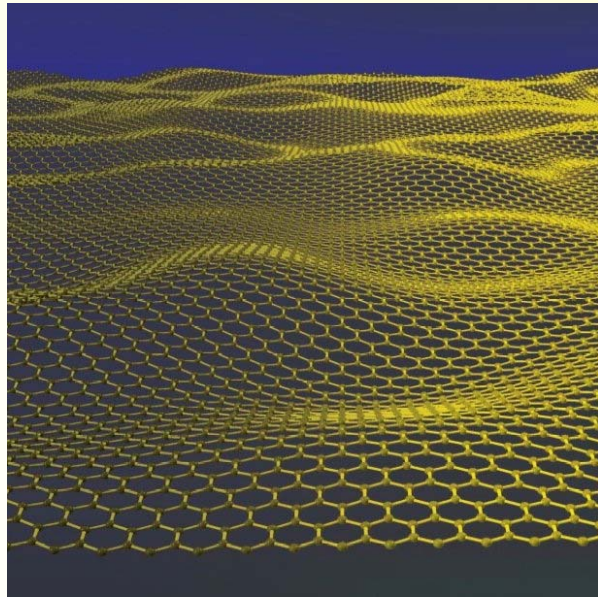
Close to the two  
Fermi points  $\mathbf{K}$ ,  $\mathbf{K}'$ :

$$H \approx v_F (\mathbf{p} - \mathbf{K}) \cdot \sigma_{\text{sublattice}}$$
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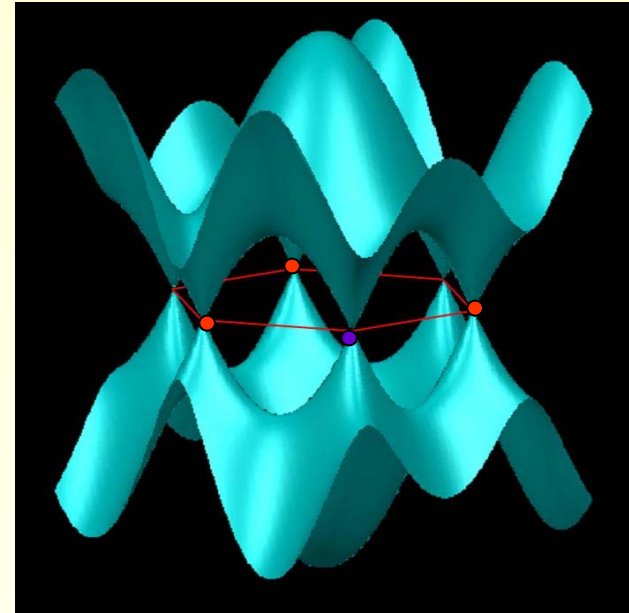
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Fermi velocity (speed of light")

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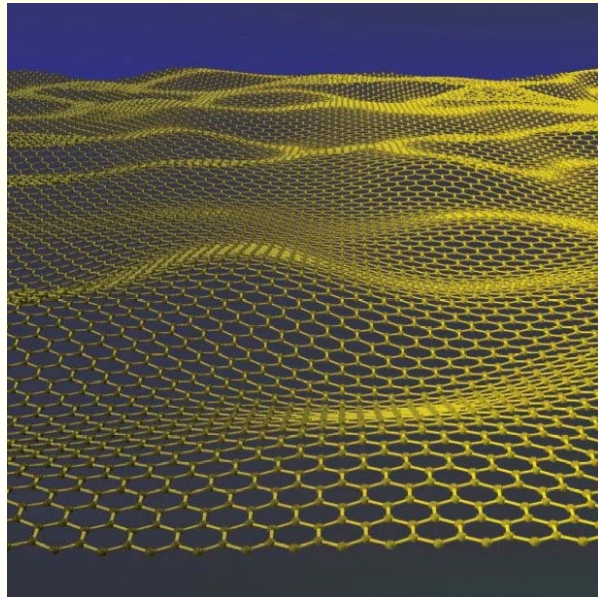
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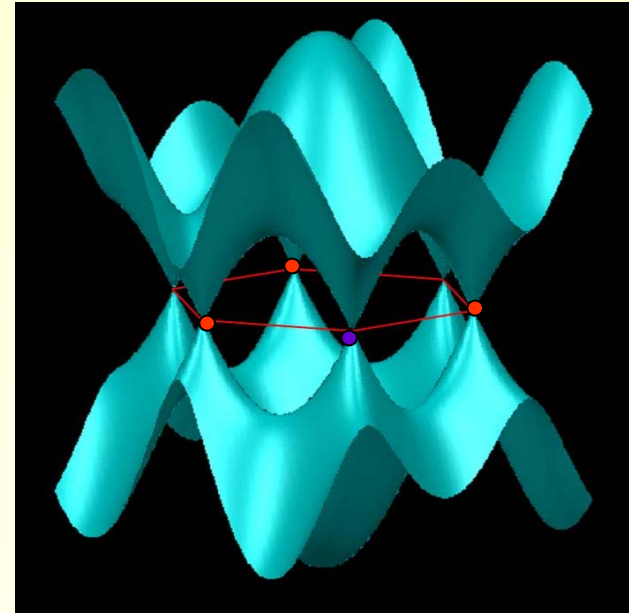
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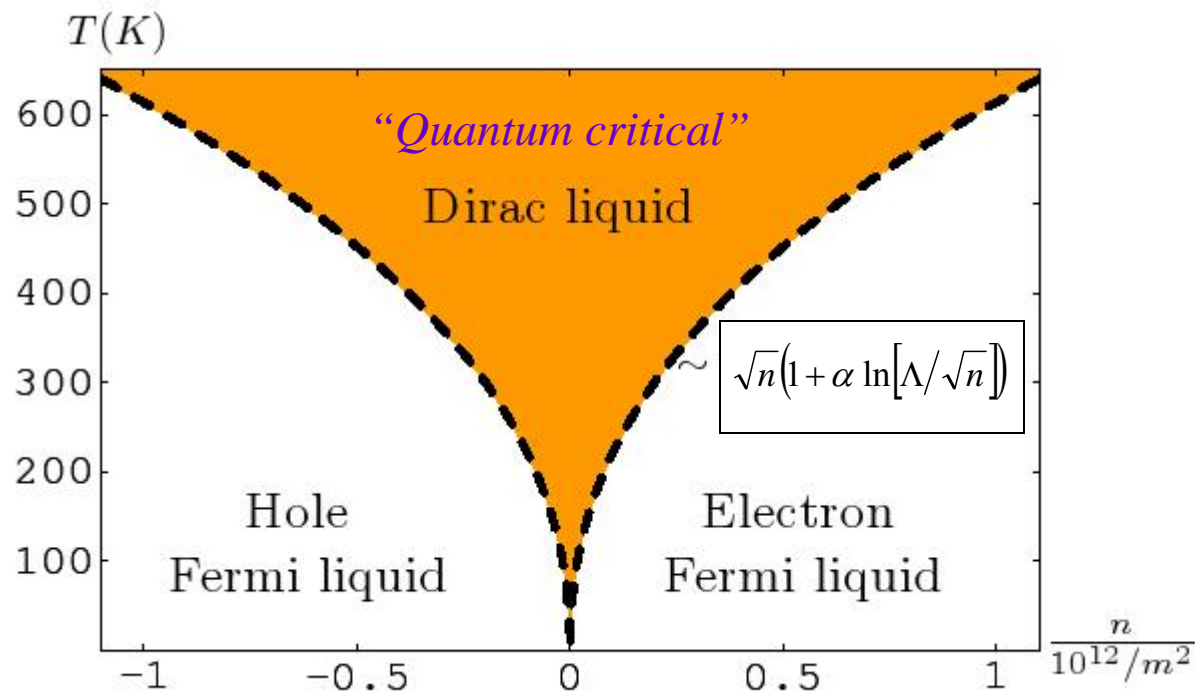
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Coulomb interactions: Fine structure constant

$$\alpha \equiv \frac{e^2}{\epsilon \hbar v_F} = O(1)$$

# Relativistic fluid at the Dirac point

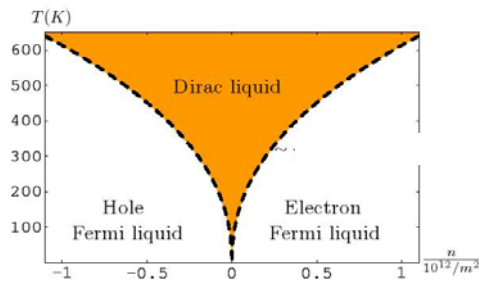
Expect relativistic plasma physics of interacting particles and holes!



*D. Sheehy, J. Schmalian, Phys. Rev. Lett. 99, 226803 (2007).*

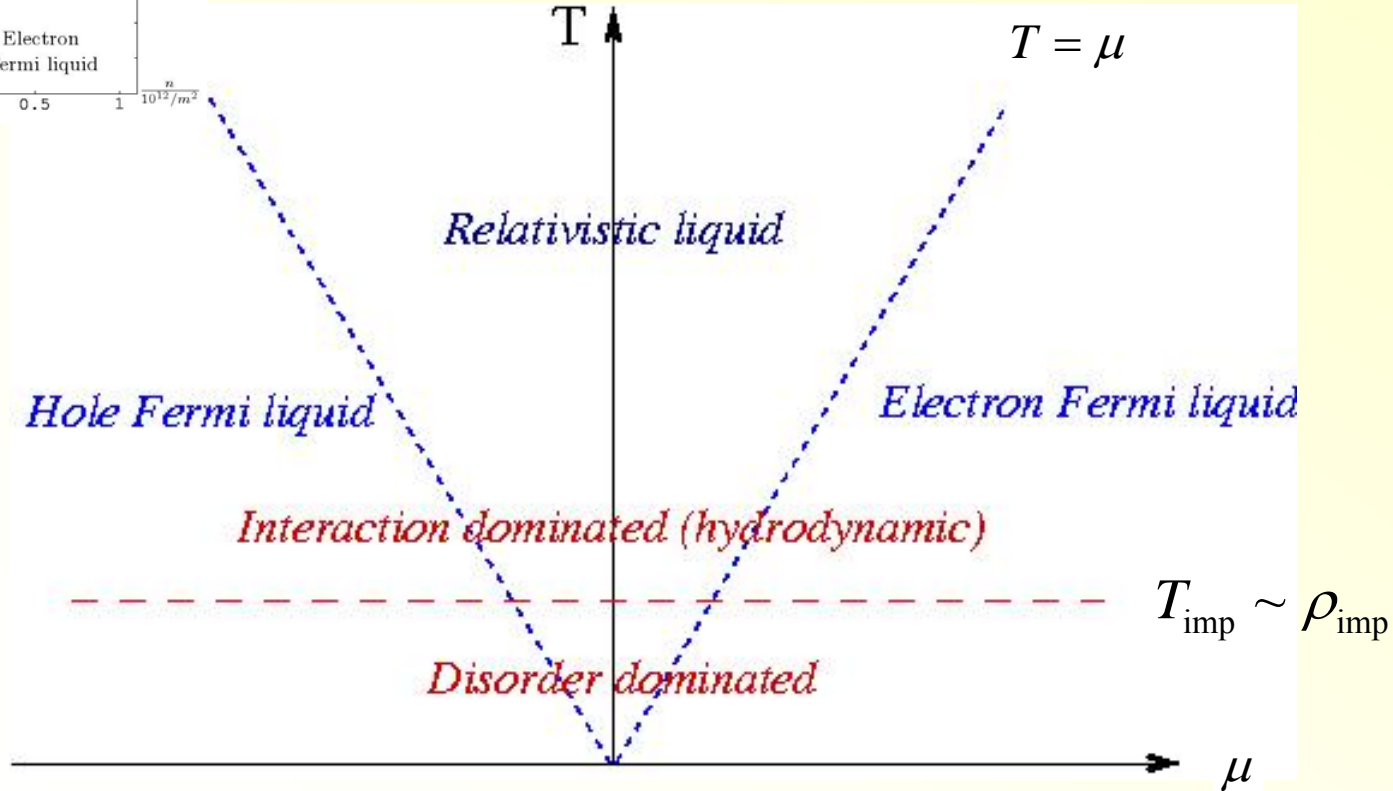
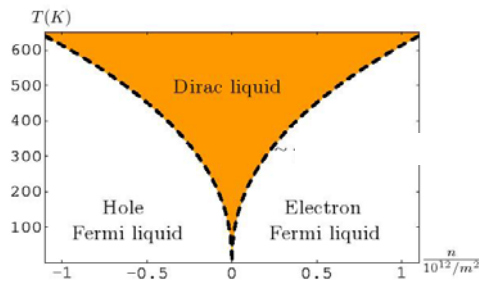
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Expect **relativistic plasma** physics of interacting particles and holes!



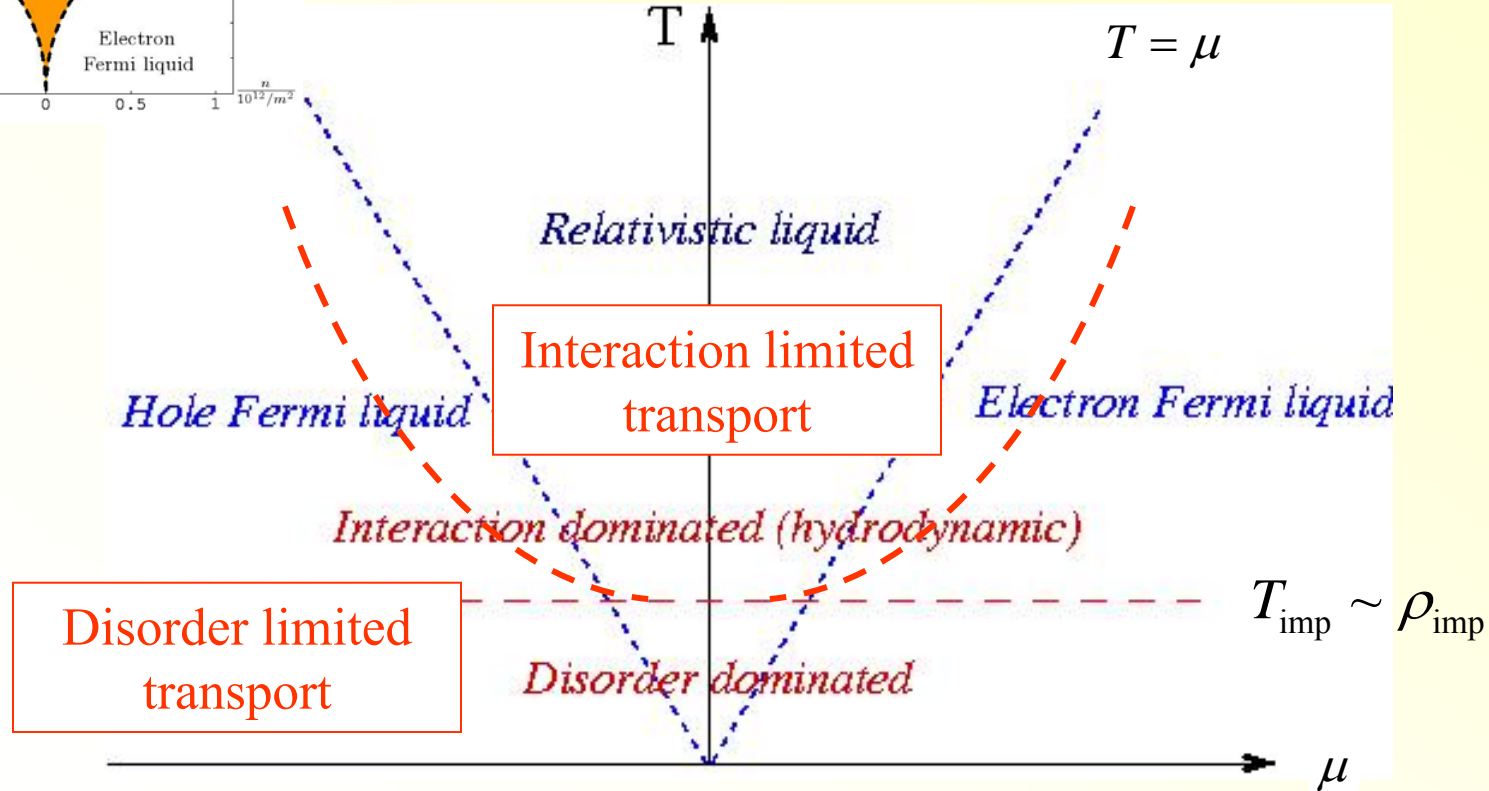
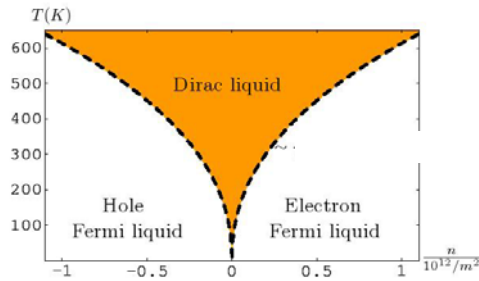
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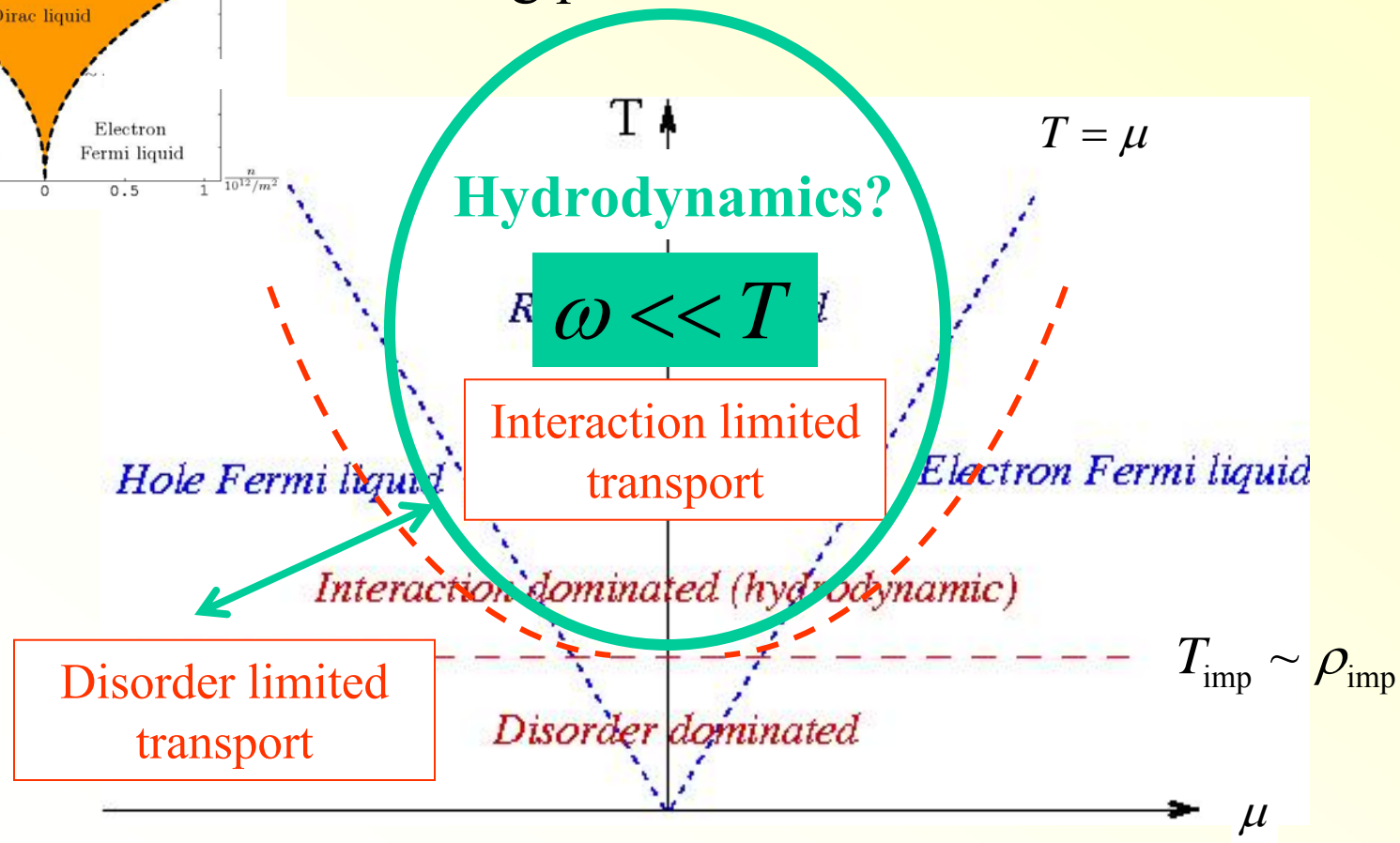
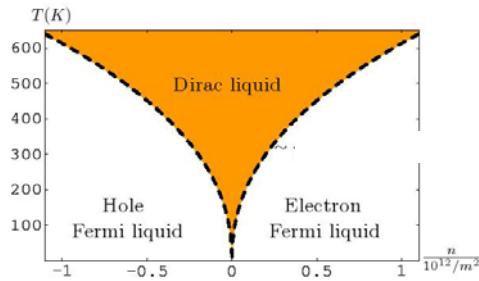
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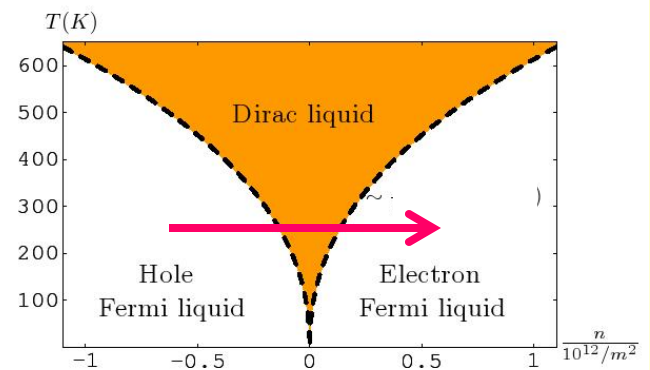
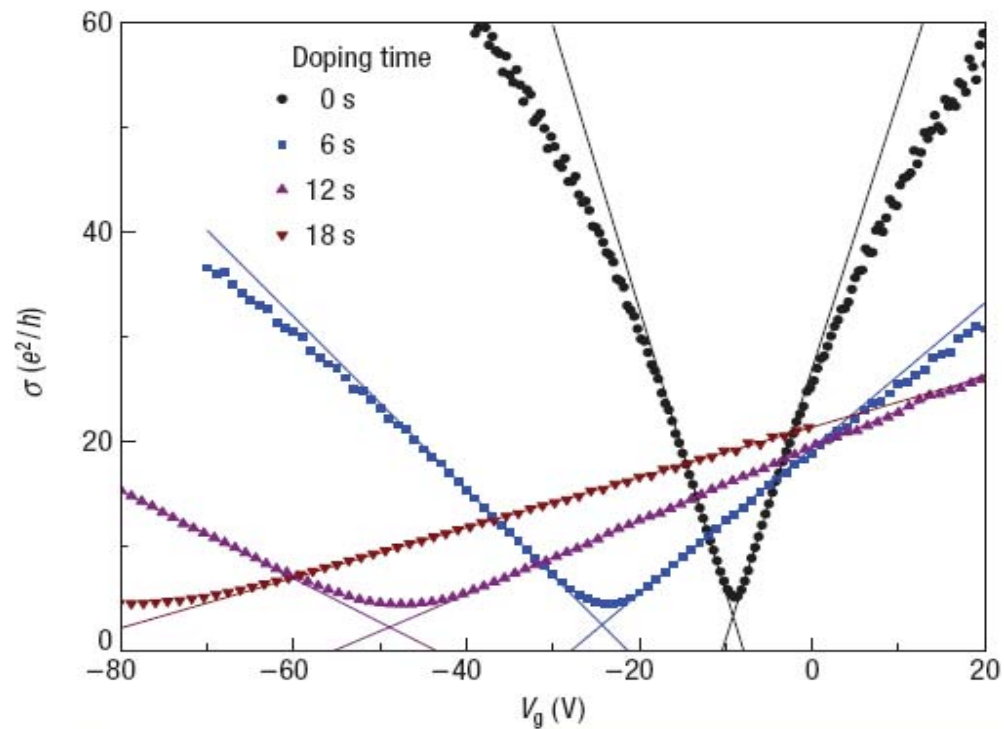


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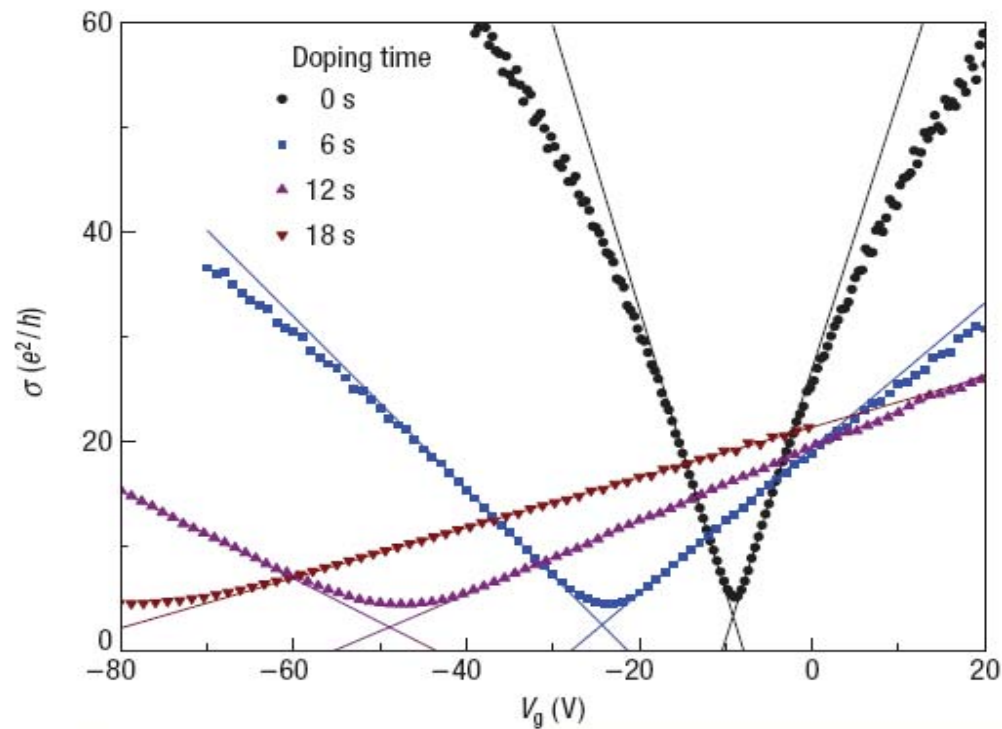


# Conductivity in and across the relativistic regime?

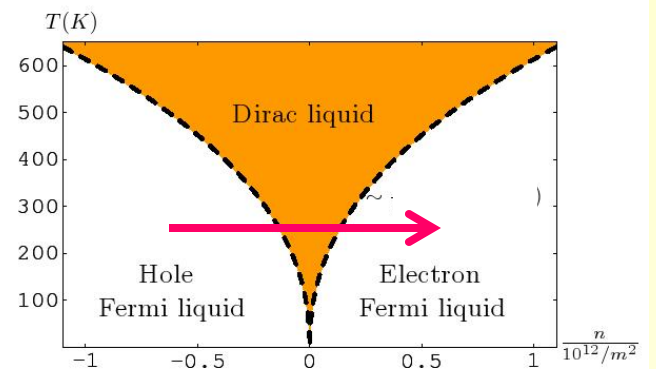


*J.-H. Chen et al. Nat. Phys. 4, 377 (2008).*

# Conductivity in and across the relativistic regime?



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+ Magnetotransport?  
e.g., Hall, Nernst effect?

# Other relativistic fluids:

- Bismuth (3d Dirac fermions with very small mass)
- Effective theories close to quantum phase transitions
- Conformal field theories  
E.g.: strongly coupled Non-Abelian gauge theories  
(QCD): treatment via AdS-CFT

# Low energy effective theory at quantum phase transitions

Relativistic effective field theories  $\leftrightarrow z = 1$ ;  
arise often due to particle-hole symmetry

Example: Superconductor-insulator transition (SIT)

*Bhaseen, Green, Sondhi (PRL '07).*

*Hartnoll, Kovtun, MM, Sachdev (PRB '07)*

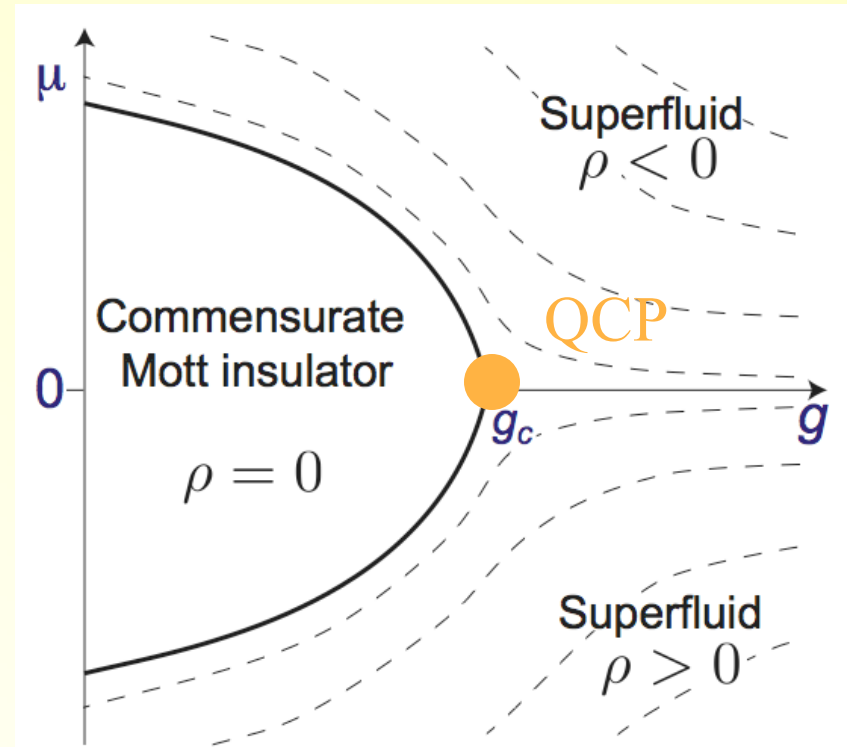
# SI-transition: Bose Hubbard model

Bose-Hubbard model

$$H = -t \sum_{\langle ij \rangle} b_j^\dagger b_i + U \sum_i n_i^2 - \mu \sum_i n_i$$

Coupling

$g \equiv \frac{t}{U}$  tunes the SI-transition



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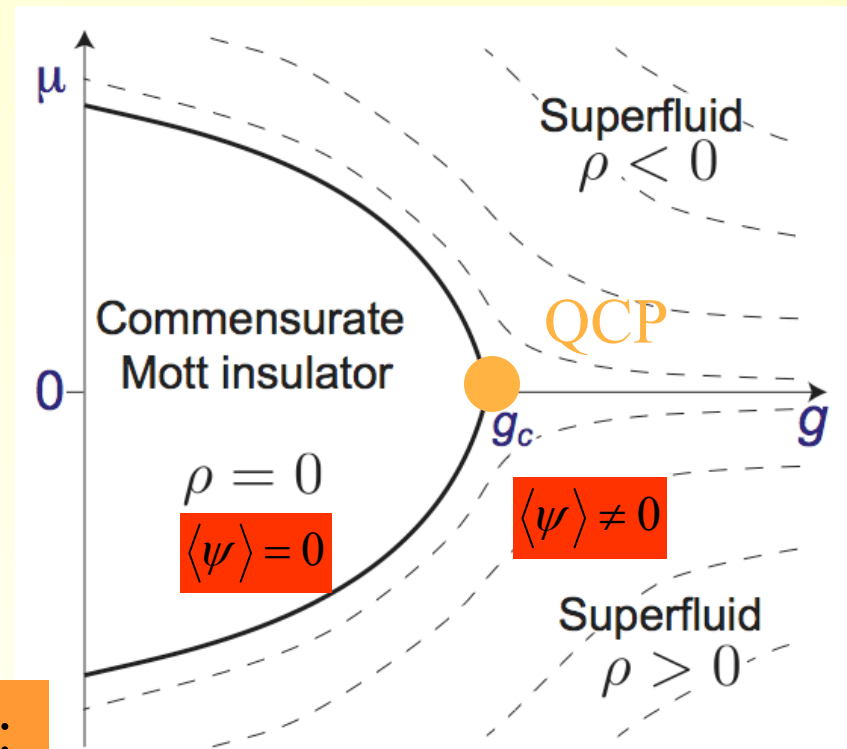
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Effective action around  $g_c$  ( $\mu = 0$ ):

$$\mathcal{S} = \int d^2r d\tau \left[ |\partial_\tau \psi|^2 + v^2 |\vec{\nabla} \psi|^2 - g |\psi|^2 + \frac{u}{2} |\psi|^4 \right]$$



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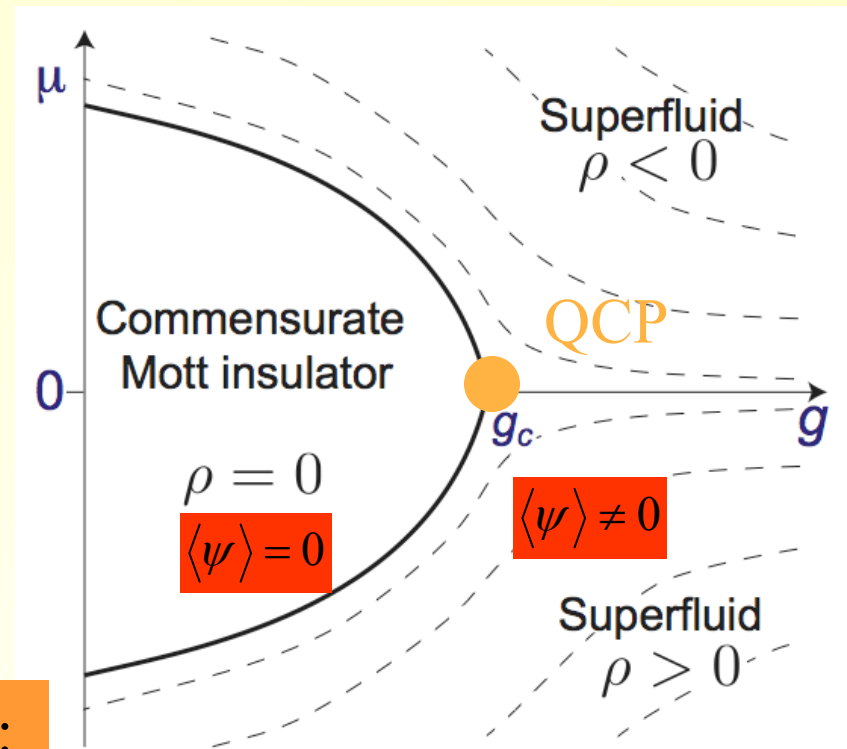
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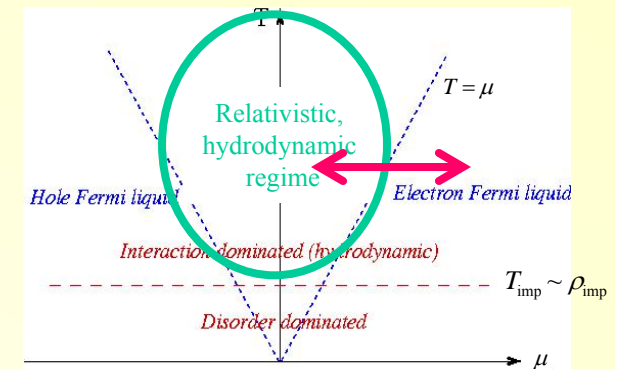
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→ Relativistic field theory in d=2+1



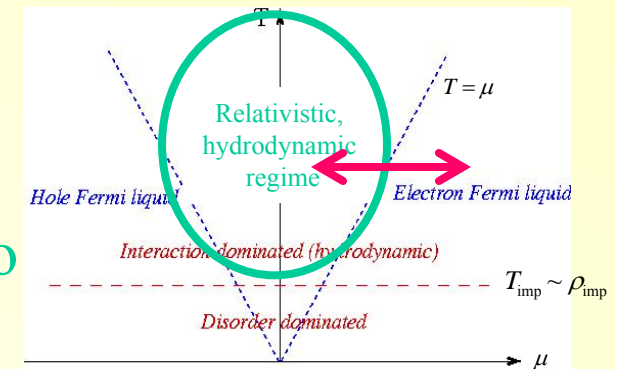
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- **Transport characteristics** of the relativistic plasma in lightly doped graphene and close to quantum criticality?



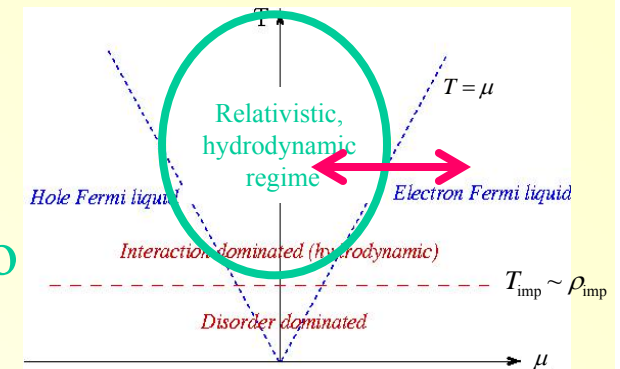
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# Questions

- **Transport characteristics** of the relativistic plasma in lightly doped graphene and close to quantum criticality?
- How does the **relativistic regime** connect to **Fermi liquid** behavior at large doping?
- What is the range of **validity of relativistic magneto-hydrodynamics**?
- **Beyond hydrodynamics**?



# Model of graphene

Graphene with Coulomb interactions and disorder

$$H = H_0 + H_1 + H_{\text{dis}}$$

Tight binding kinetic energy

$$H_0 = - \sum_{a=1}^N \int d\mathbf{x} \left[ \Psi_a^\dagger \left( i v_F \vec{\sigma} \cdot \vec{\nabla} + \mu \right) \Psi_a \right]$$

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RG:

$$\frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3)$$

$$\alpha(T) = \frac{\alpha^0}{1 + (\alpha^0/4) \ln(\Lambda/T)} \stackrel{T \rightarrow 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$

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Disorder: charged impurities

$$H_{\text{dis}} = \int d\mathbf{x} V_{\text{dis}}(\mathbf{x}) \Psi_a^\dagger(\mathbf{x}) \Psi_a(\mathbf{x}) \quad V_{\text{dis}}(\mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{x}_i) \frac{Z e^2}{\epsilon|\mathbf{x} - \mathbf{x}_i|}$$

# Time scales

*MM, L. Fritz, and S. Sachdev, cond-mat 0805.1413.*

1. Inelastic scattering rate  
(Electron-electron interactions)

$$\tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \frac{1}{\max[1, \mu/T]}$$

Relativistic regime ( $\mu < T$ ):  
Relaxation rate set by temperature,  
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3. Deflection rate due to magnetic field  
(Cyclotron frequency of non-interacting  
particles with typical thermal energy )

$$\tau_B^{-1} \sim \omega_c^{\text{typ}} \sim \frac{eBv_F^2}{\max[T, \mu]}$$

# Regimes

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3. Disorder limited transport  
(inelastic scattering ineffective due to  
nearly conserved momentum)

$$\mu \gg T$$

$$\tau_{ee}^{-1} \gtrsim \tau_{imp}^{-1}$$

# Hydrodynamic Approach

# Hydrodynamics

Hydrodynamic collision-dominated regime

Long times,  
Large scales

$$t \gg \tau_{ee}$$

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$$t \gg \tau_{ee}$$


- Local equilibrium established:  $T_{loc}(r), \mu_{loc}(r); \vec{u}_{loc}(r)$
- Study relaxation towards global equilibrium
- Slow modes: Diffusion of the density of conserved quantities:
  - Charge
  - Momentum
  - Energy

# Relativistic Hydrodynamics

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B 76, 144502 (2007).*

Energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

Current 3-vector

$$J^\mu = \rho u^\mu + \nu^\mu$$

$$\begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \end{pmatrix}$$

$u^\mu$  : Energy velocity:  $u^\mu = (1,0,0) \rightarrow$  No energy current

$\nu^\mu$  : Dissipative current (“heat current”)

$\tau^{\mu\nu}$  : Viscous stress tensor (Reynold’s tensor)

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+ Thermodynamic relations

$$\varepsilon + P = Ts + \mu\rho, \quad d\varepsilon = Tds + \mu d\rho,$$

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Energy/momentum conservation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu} \delta_{\mu 0}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

$$\vec{E} = -i\vec{k} \frac{2\pi}{|k|} \rho_{\vec{k}} \quad \text{Coulomb interaction}$$

# Relativistic Hydrodynamics

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B 76, 144502 (2007).*

$$J^\mu = \rho u^\mu + v^\mu$$

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

Conservation laws (equations of motion):

$$\partial_\mu J^\mu = 0 \quad \text{Charge conservation}$$

Energy/momentum conservation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu} \delta_{\mu 0}$$

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Coulomb interaction

Heat current and viscous tensor?



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(Second law):

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$$\Rightarrow v^\mu = \text{const.} \times A^\mu (\partial T, \partial \mu, \partial u; F^{\mu\nu})$$
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$$v^\mu = \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[ (-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T} \right]$$

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**B small!**

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Irrelevant for response at  $k \rightarrow 0$

One single transport coefficient (instead of two)!

# Meaning of $\sigma_Q$ ?

- Dimension of electrical conductivity
- At zero doping (particle-hole symmetry):

$$\sigma_Q = \sigma_{xx} (\rho_{\text{imp}} = 0)$$

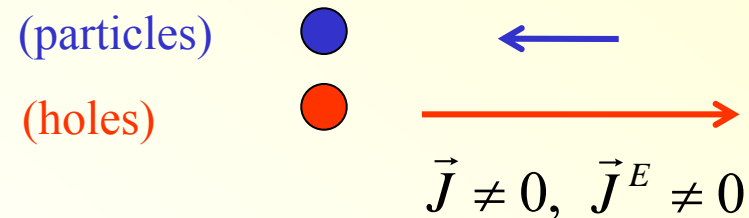
= Universal d.c. conductivity of the pure system

Why is  $\sigma_{xx} (\rho_{\text{imp}} = 0)$  finite ??

# Universal conductivity $\sigma_Q$

Standard situation: No particle-hole symmetry ( $\rho \neq 0$ )

- Current is carried predominantly by majority carriers
- Finite current implies finite momentum:



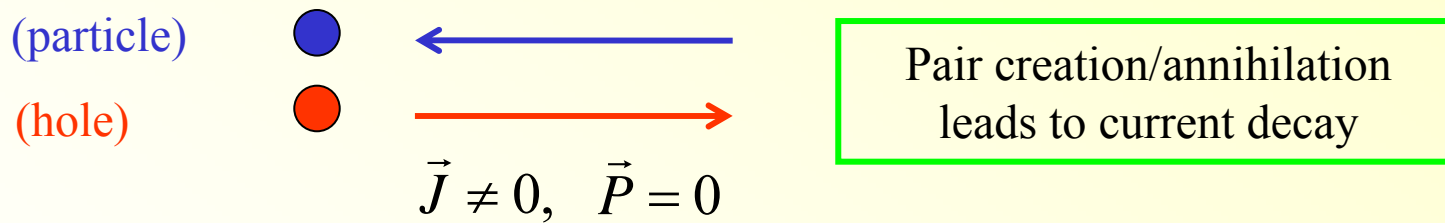
- In the absence of impurities:  
Momentum conservation implies **infinite conductivity!**

# Universal conductivity $\sigma_0$

*K. Damle, S. Sachdev, (1996).*

Particle-hole symmetry ( $\rho = 0$ )

- Key: Charge current without momentum (energy current)!



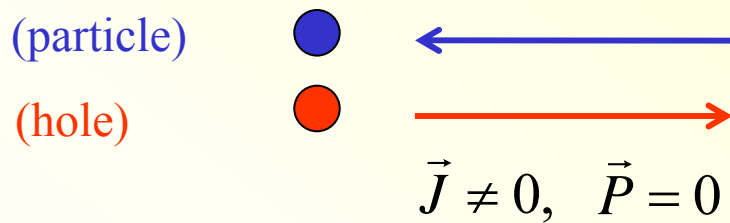
- Finite “quantum critical” conductivity!

# Universal conductivity $\sigma_0$

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Quantum critical situation: Particle-hole symmetry ( $\rho = 0$ )

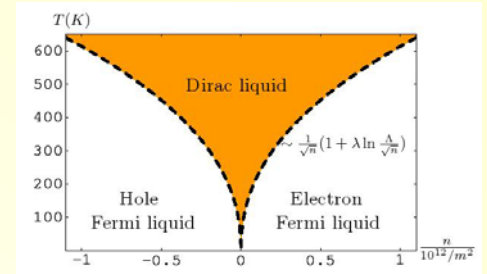
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Pair creation/annihilation  
leads to current decay

- Finite “quantum critical” conductivity!
- As in quantum criticality:  
Relaxation time set by temperature alone

$$\tau_{ee} \approx \frac{\hbar}{\alpha^2 k_B T}$$

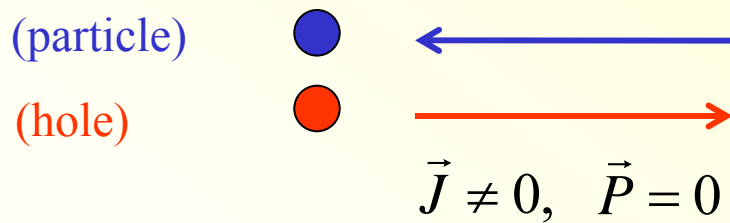


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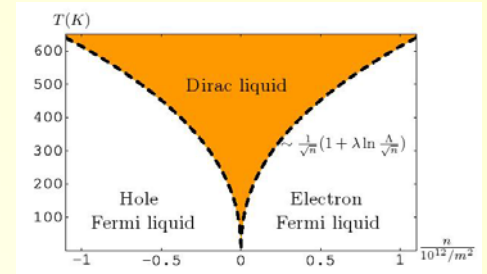
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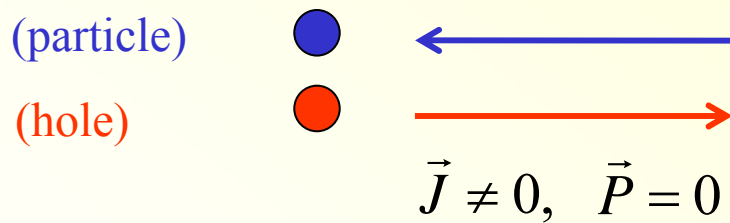
$$\sigma_{Drude} = \frac{e}{m} \rho \tau \rightarrow \sigma_Q \sim \frac{e}{k_B T / v^2} \left( e \frac{(k_B T)^2}{(\hbar v)^2} \right) \frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2} \frac{e^2}{h}$$

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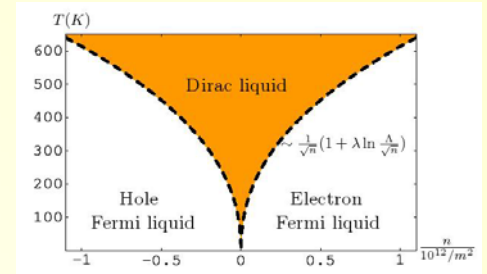
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Exact (Boltzmann)

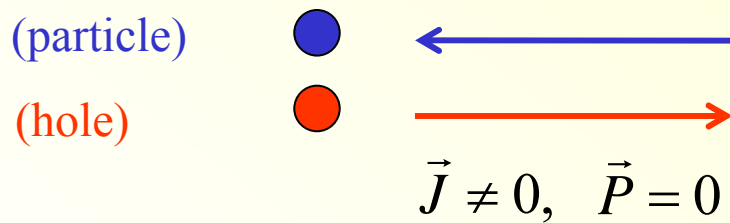
$$\sigma_Q(\mu = 0) = \frac{0.76 e^2}{\alpha^2 h}$$

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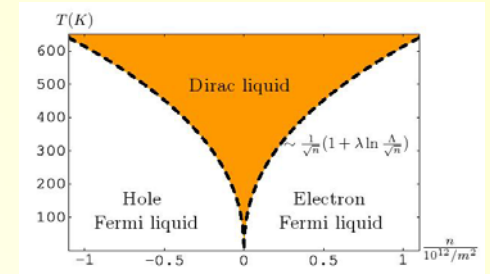
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Marginal irrelevance of Coulomb:

$$\alpha \approx \frac{4}{\log(\Lambda/T)}$$

# Thermoelectric response

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B 76, 144502 (2007).*

Charge and heat current:

$$J^\mu = \rho u^\mu - v^\mu$$

$$Q^\mu = (\varepsilon + P)u^\mu - \mu J^\mu$$

Thermo-electric response

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} \\ T\hat{\alpha} & \hat{\kappa} \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla}T \end{pmatrix}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

etc.

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- i) Solve linearized hydrodynamic equations
- ii) Read off the response functions (*Kadanoff & Martin 1960*)

# Results from Hydrodynamics

# Response functions at B=0

$$\text{Symmetry } z \rightarrow -z : \quad \sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$$

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B = 0) = \left( \sigma_Q + \frac{\rho^2}{P + \epsilon} \frac{\tau}{1 - i\omega\tau} \right)$$

Universal conductivity at the quantum critical point  $\rho = 0$

Drude-like conductivity, divergent for  
Momentum conservation ( $\rho \neq 0$ )!

$$\tau \rightarrow \infty, \omega \rightarrow 0, \rho \neq 0$$

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Coulomb correction  
( $g = 2\pi e^2$ )


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Coulomb correction  
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Thermal conductivity:

$$\kappa_{xx}(\omega, k; B = 0) = \sigma_Q \frac{\mu^2}{T} + \frac{s^2 T}{P + \varepsilon} \frac{\tau}{1 - i\omega\tau} + \mathcal{O}(k^2).$$

Relativistic Wiedemann-Franz-like  
relations between  $\sigma$  and  $\kappa$  in the quantum  
critical window!

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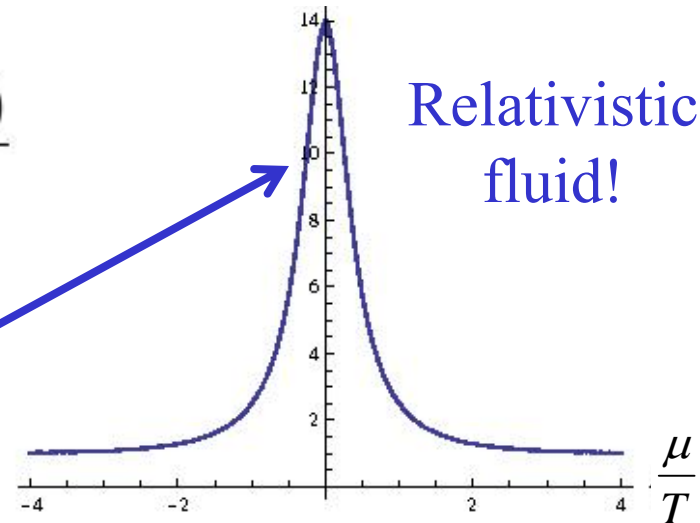
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Coulomb correction  
( $g = 2\pi e^2$ )

Thermopower:

$$\alpha_{xx}(\mu, \omega = 0) = -\frac{\pi^2}{3e} k_B^2 T \frac{d\sigma(\mu, \omega = 0)}{d\mu}$$

$$-\frac{3e}{\pi^2} \frac{1}{k_B^2 T} \frac{\alpha_{xx}}{d\sigma_{xx}/d\mu}$$



Only valid in the **Fermi liquid** regime,  
but violated in the **relativistic window**.

# B > 0 : Cyclotron resonance

E.g.: Longitudinal conductivity

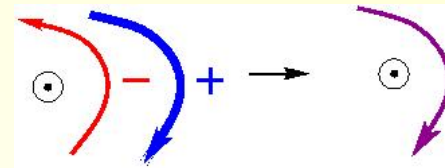
$$\sigma_{xx}(\omega) = \sigma_Q \frac{\omega (\omega + i\gamma + i\omega_c^2/\gamma)}{(\omega + i\gamma)^2 - \omega_c^2}$$

Poles in the response

$$\omega = \pm \omega_c^{\text{QC}} - i\gamma - i/\tau$$

Collective cyclotron frequency of the relativistic plasma

$$\omega_c^{\text{QC}} = \frac{\rho B}{(\varepsilon + P)/v_F^2} \leftrightarrow \omega_c^{\text{FL}} = \frac{e B}{m}$$



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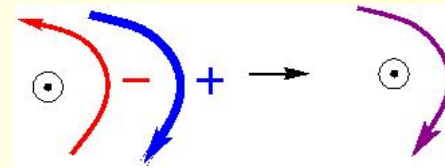
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Intrinsic, interaction-induced broadening

( $\leftrightarrow$  Galilean invariant systems:

No broadening due to Kohn's theorem)

$$\gamma = \sigma_Q \frac{B^2}{(\varepsilon + P)/v_F^2}$$

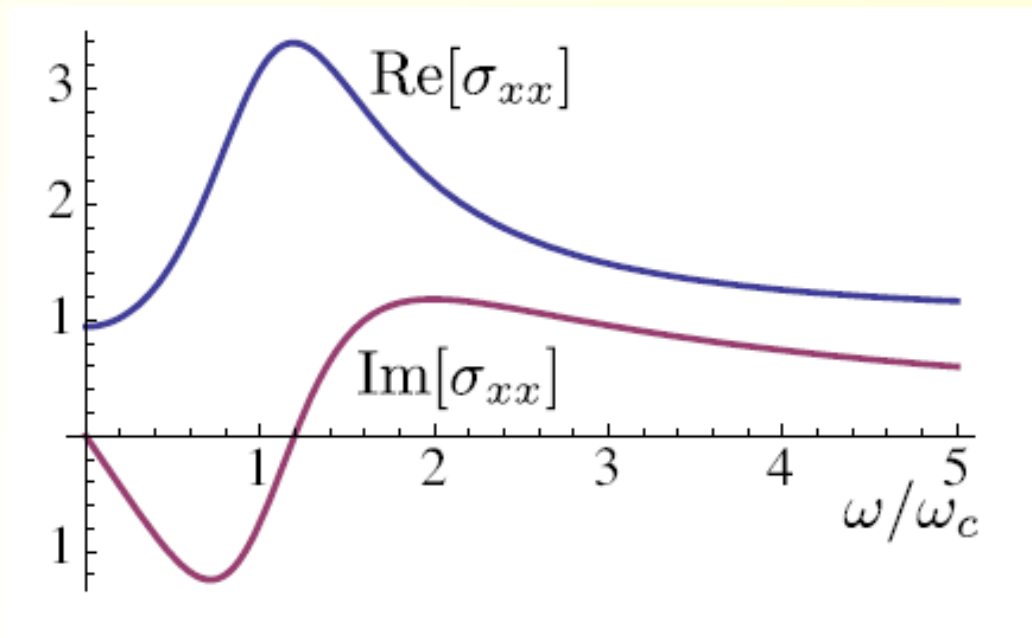
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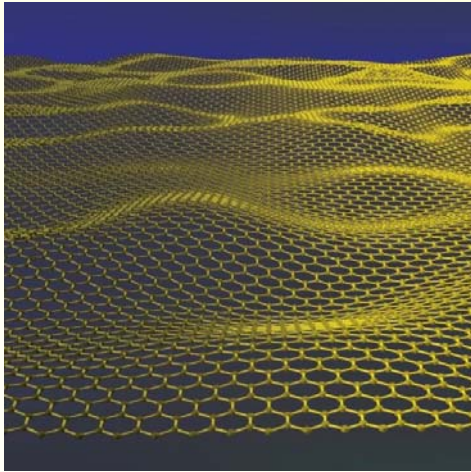
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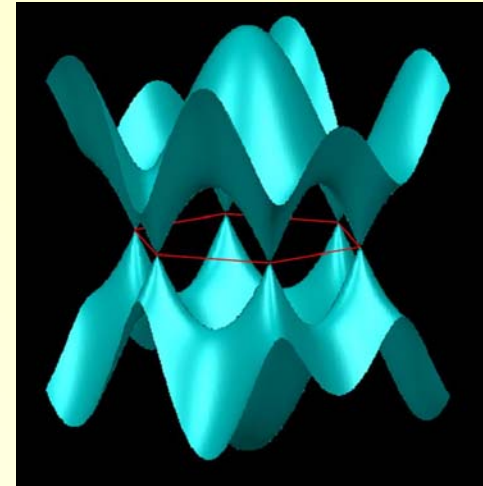


# Can the resonance be observed?



$$\omega = \pm\omega_c - i\gamma - i/\tau$$

$$v_F = 1.1 \cdot 10^6 \text{ m/s} \\ \approx c/300$$



## Conditions to observe collective cyclotron resonance

Collision-dominated regime

$$\hbar\omega_c \ll \alpha^2 k_B T$$

Small broadening

$$\gamma, \tau^{-1} < \omega_c$$

Quantum critical regime

$$\rho \leq \rho_{th} = \frac{(k_B T)^2}{(\hbar v_F)^2}$$

High T: no Landau quantization

$$E_{LL} = \hbar v_F \sqrt{\frac{2eB}{\hbar c}} \ll k_B T$$

Parameters:

$$T \approx 300 \text{ K} \\ B \approx 0.1 \text{ T} \\ \rho \approx 10^{11} \text{ cm}^{-2} \\ \omega_c \approx 10^{13} \text{ s}^{-1}$$

# Does relativistic hydrodynamics apply?

- Do  $T$  and  $\mu$  not break relativistic invariance?
- Validity at large chemical potential?
- Beyond linearization in magnetic field?
- Treatment of disorder?

# Boltzmann Approach

*MM, L. Fritz, and S. Sachdev, cond-mat 0805.1413.*

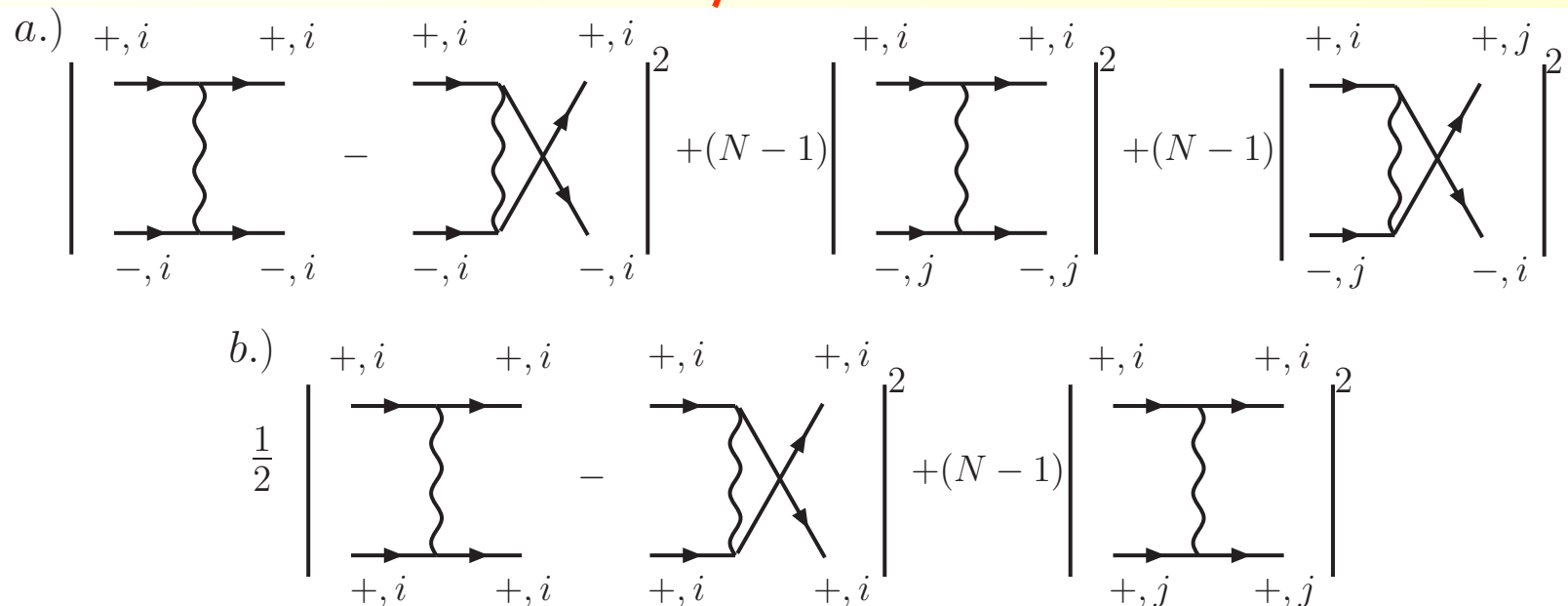
- Recover and refine the hydrodynamic description
- Describe relativistic-to-Fermi-liquid crossover
- Go beyond hydrodynamics

# $\sigma_Q$ from Boltzmann

*L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289*

Boltzmann equation in Born approximation

$$\left( \partial_t + e[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot \frac{\partial}{\partial \mathbf{k}} \right) f_{\pm}(\mathbf{k}, t) = \alpha^2 I_{\text{coll}}^{Cb}[\mathbf{k}, t | \{f_{\pm}(\mathbf{k}', t)\}] + \Delta I_{\text{coll}}^{dis}[\mathbf{k}, t | \{f_{\pm}(\mathbf{k}', t)\}]$$



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Linearization:

$$f_{\pm}(\mathbf{k}, t) = f_{\pm}^{eq}(\mathbf{k}, t) + \delta f_{\pm}(\mathbf{k}, t)$$

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Great simplification: Divergence of forward scattering amplitude in 2d

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# $\sigma_Q$ from Boltzmann

*L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289*

Boltzmann equation in Born approximation

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$$\sigma_Q(\mu = 0) \approx \frac{0.76 e^2}{\alpha^2 h}$$

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$$\begin{aligned} f_{\lambda}(\mathbf{r}, \mathbf{k}, \omega) = & 2\pi\delta(\omega) f_{\lambda}^0(k, T(\mathbf{r})) \\ & + f_{\lambda k}^0 [1 - f_{\lambda k}^0] \frac{v_F}{T^2} \mathbf{e}_{\mathbf{k}} \cdot \left[ e\mathbf{E}(\omega) g_{\parallel, \lambda}^{(E)} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g_{\parallel, \lambda}^{(T)} \left( \frac{v_F k}{T}, \omega \right) \right] \\ & + f_{\lambda k}^0 [1 - f_{\lambda k}^0] \frac{v_F}{T^2} (\mathbf{e}_{\mathbf{k}} \times \mathbf{e}_z) \cdot \left[ \mathbf{E}(\omega) g_{\perp, \lambda}^{(E)} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g_{\perp, \lambda}^{(T)} \left( \frac{v_F k}{T}, \omega \right) \right] \end{aligned}$$

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Momentum or energy-current mode

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$$\sum_{\lambda} \int d^2 k f_{\lambda k}^0 (1 - f_{\lambda k}^0) \phi_{n \geq 2}(\lambda, k) \phi_{0,1}(\lambda, k) = 0.$$

Relativistic dispersion ensures that  $\phi_0$  only couples to  $\phi_1$  for clean systems!

# Conductivity: $\sigma_Q$

*L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289*

General doping:

Clean system:

$$\sigma_{xx}(\omega; \mu, \Delta = 0) = e^2 \frac{\rho^2 v_F^2}{\varepsilon + P} \frac{1}{(-i\omega)} + \sigma_Q.$$

Precise expression  
for  $\sigma_Q$ !

$$\sigma_Q(\mu, \omega) = \frac{e^2}{\hbar} \frac{1}{\alpha^2} \frac{2\hat{g}_1}{N} \left[ I_+^{(1)} - \frac{\rho^2 (\hbar v)^2}{(\varepsilon + P)T} \right]^2 \frac{1}{1 - i\omega\tau_{ee}}$$

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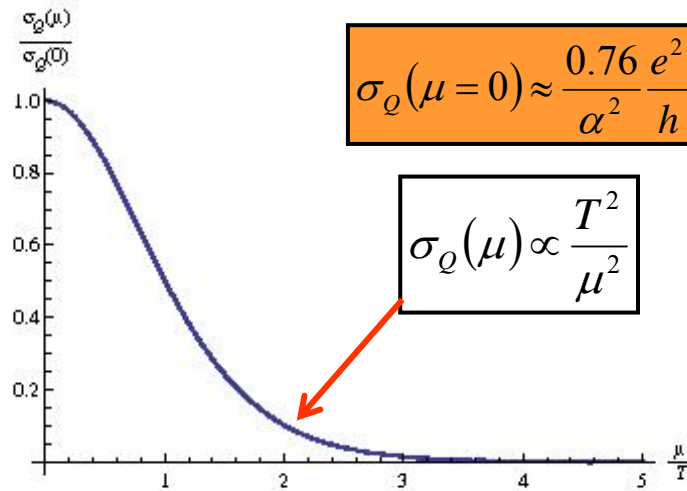
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Gradual disappearance  
of relativistic physics



Will appear in all Boltzmann formulae below!

# Conductivity: crossover

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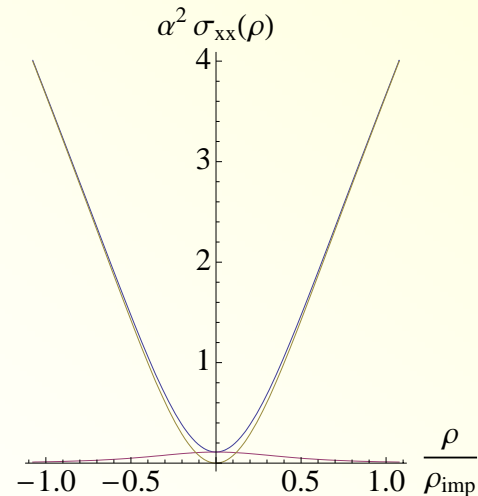
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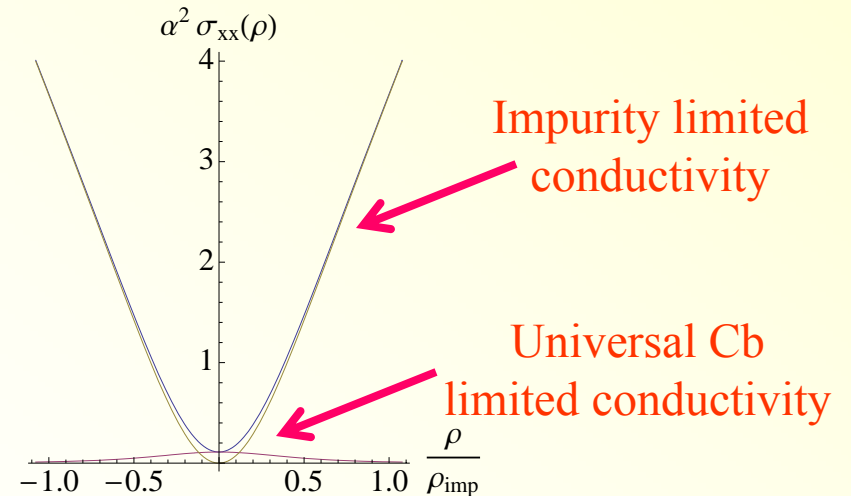
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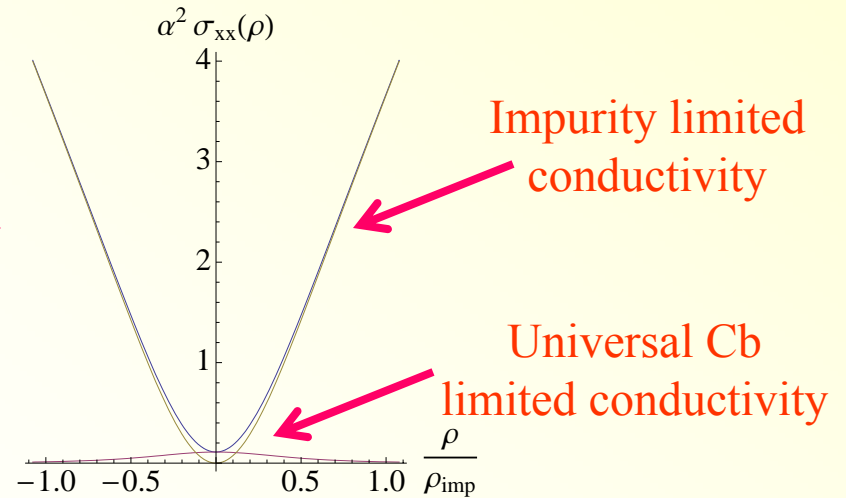
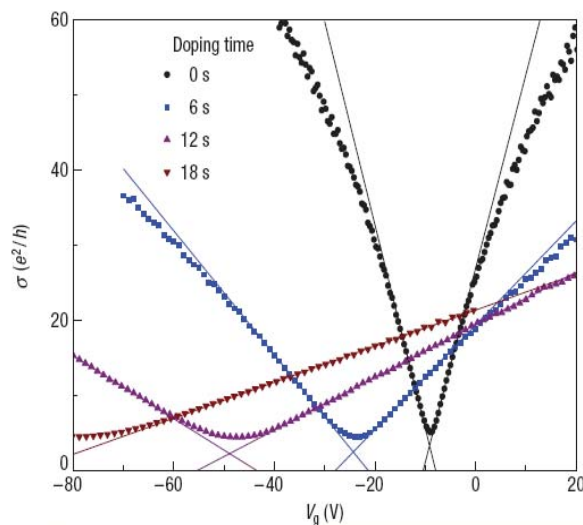
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*J.-H. Chen et al. Nat. Phys. 4, 377 (2008).*



# Magnetotransport

- **Strategy:** describe the **slow dynamics of the momentum mode  $\varphi_0$**  in very weak disorder and moderate magnetic field

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- **At small  $B$ ,** one transport coefficient is subdominant  
→ Relativistic hydrodynamics with **only one transport coefficient  $\sigma_Q$**  is recovered!

$$\tau_{ee}^{-1} \gg \tau_B^{-1}$$

$$\sigma_{xx}(\omega, B) = \sigma_{xx}^{\text{MHD}}(\omega, B) + \mathcal{O}(b/\alpha^2, \omega/\alpha^2)$$

Corrections to hydrodynamics

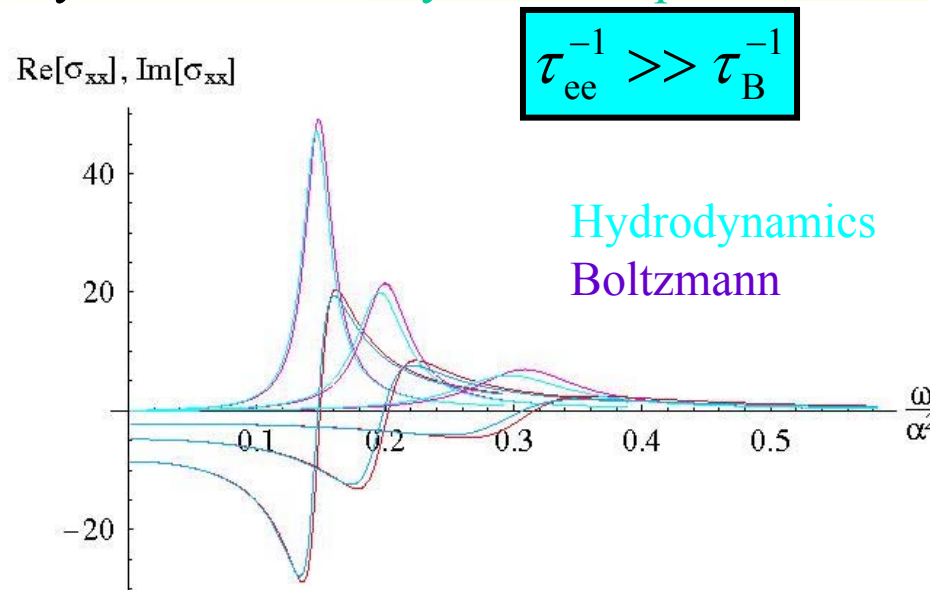
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**Cyclotron  
resonance:**



# Cyclotron resonance revisited

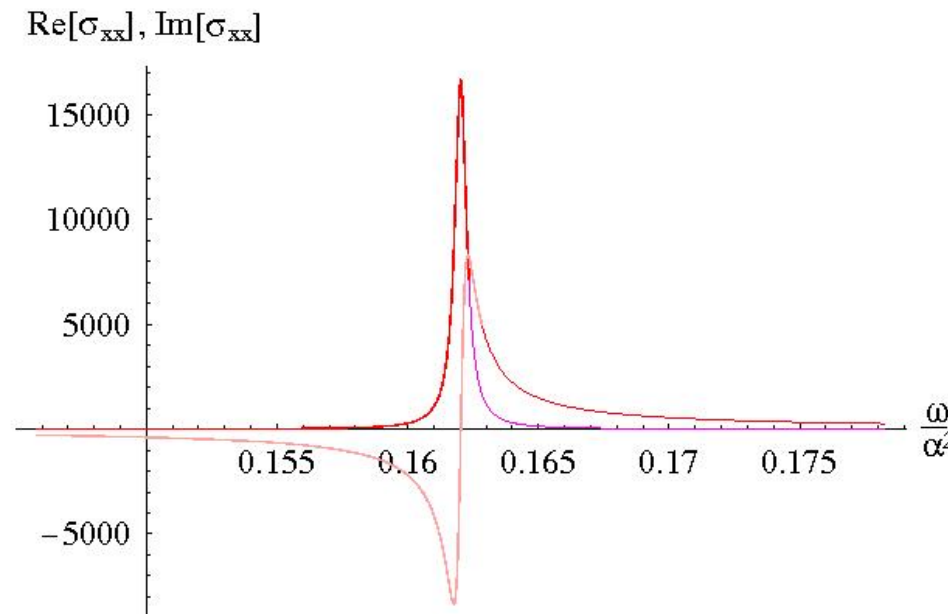
## Crossover to Fermi liquid regime:

- Semiclassical  $\omega_c$  recovered at  $\mu \gg T$
- Broadening goes to zero - **Kohn's theorem recovered**: Non-broadening of the resonance for a single parabolic band.

$$\omega_c^{(0)} = \frac{\rho B}{\varepsilon + P} \rightarrow \frac{eB}{\mu/v_F^2} = \frac{eB}{\hbar k_F/v_F}$$

$$\gamma \equiv \frac{\sigma_Q B^2 v_F^2}{(\varepsilon + P)}$$

$$\gamma \propto \sigma_Q(\mu) \xrightarrow{\mu \gg T} 0$$



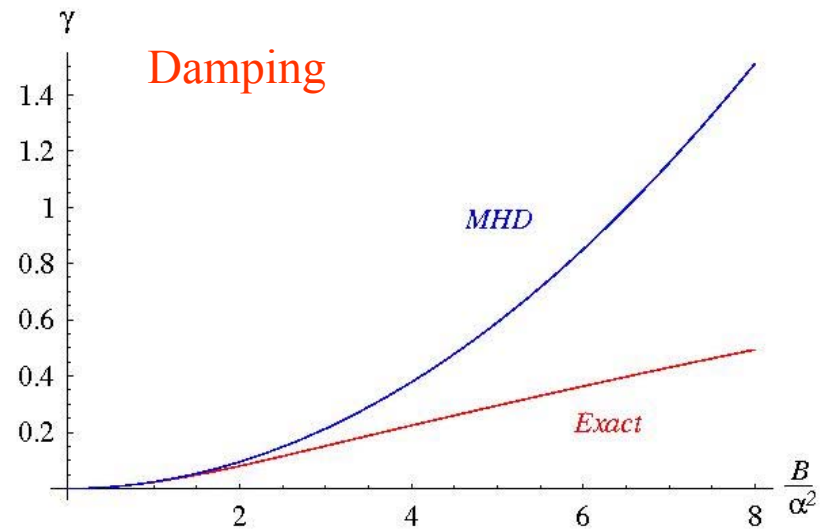
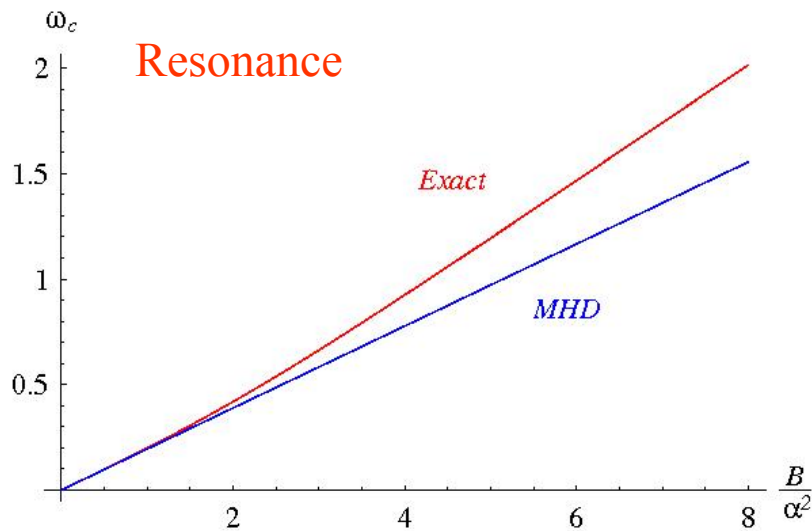
# Cyclotron resonance revisited

Beyond hydrodynamics: Towards ballistic magnetotransport

$$\mu = T$$

Large fields

$$\tau_B^{-1} > \tau_{ee}^{-1} \gg \tau_{imp}^{-1}, \omega$$



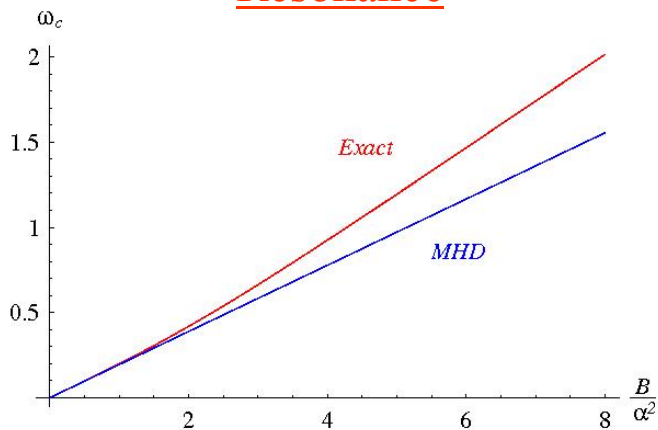
# Strongly coupled liquids

Same trends as in exact (AdS-CFT) results for strongly coupled relativistic fluids!

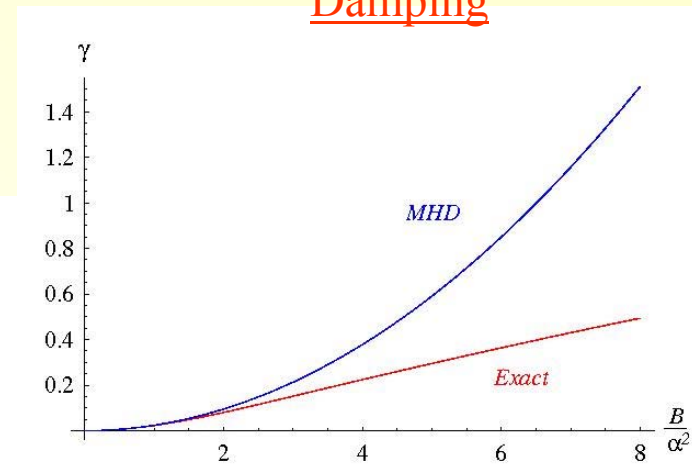
*S. Hartnoll, C. Herzog (2007)*

## Graphene

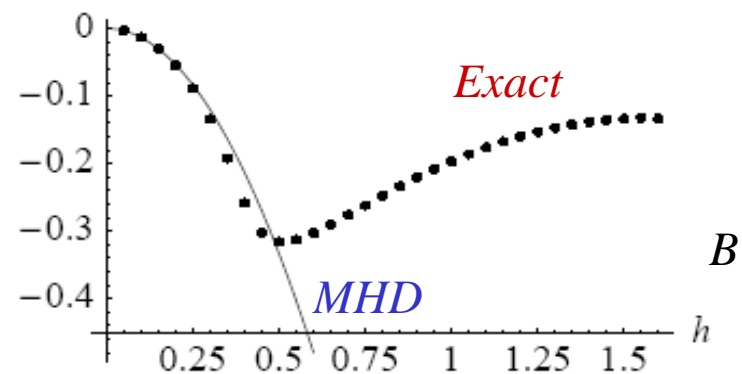
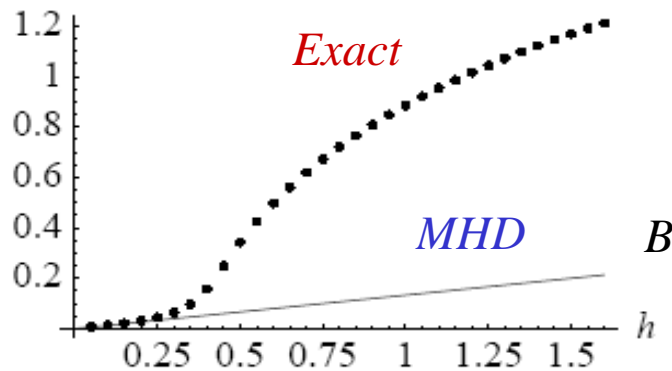
### Resonance



### Damping



$\mathcal{N}=4$  SUSY SU(N) gauge theory [flows to CFT at low energy]



# Summary

- Relativistic physics in graphene and quantum critical systems

- Hydrodynamic description:

→ collective cyclotron resonance in the relativistic regime

→ covariance: 6 frequency dependent response functions given by thermodynamics and *only one* parameter  $\sigma_Q$ .

- Boltzmann approach

→ Confirmed and refined hydrodynamic description

→ Understood relativistic-to-Fermi liquid crossover:

- From universal Coulomb-limited to disorder-limited linear conductivity in graphene
- From collective-broadened to semiclassical sharp cyclotron resonance

→ Beyond hydrodynamics: describe large fields and disorder

