Relativistic magnetotransport in graphene

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Landau Memorial Conference June 26, 2008
Outline

• Relativistic physics in graphene, quantum critical systems and conformal field theories
  → Relativistic signatures in magnetotransport: el.+th. conductivity, Peltier, Nernst effect etc.

• Hydrodynamic description
  → Collective, collision-broadened cyclotron resonance

• Boltzmann equation
  → Recover and refine hydrodynamics with Boltzmann
  → Describe relativistic-to-Fermi liquid crossover
  → Go beyond hydrodynamics
Dirac fermions in graphene

(Haldane ‘88, Semenoff ‘84)

Honeycomb lattice of C atoms
Dirac fermions in graphene
(Semenoff ’84, Haldane ‘88)

Honeycomb lattice of C atoms

Tight binding dispersion

2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

Close to the two Fermi points $K, K'$:

$$H \approx v_F (p - K) \cdot \sigma_{\text{sublattice}}$$

$$\rightarrow \quad E_k = v_F |k - K|$$
Dirac fermions in graphene

(Semenoff ’84, Haldane ’88)

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2 massless Dirac cones in the Brillouin zone:
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Fermi velocity (speed of light”)

$E_k = v_F |k - K|$
Dirac fermions in graphene

(Honeycomb lattice of C atoms)

Tight binding dispersion

2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

Fermi velocity (speed of light”)

Coulomb interactions: Fine structure constant

\[ H \approx v_F \left( p - K \right) \cdot \sigma_{\text{sublattice}} \]

\[ E_k = v_F \left| k - K \right| \]

\[ v_F \approx 1.1 \cdot 10^6 \text{ m/s} \approx \frac{c}{300} \]

\[ \alpha \equiv \frac{e^2}{\varepsilon \hbar v_F} = O(1) \]
Relativistic fluid at the Dirac point

Expect relativistic plasma physics of interacting particles and holes!

\[ \sqrt{n} \left( 1 + \alpha \ln \left[ \frac{\Lambda}{\sqrt{n}} \right] \right) \]

\[ n \sim \Omega^+ \ln \left( \alpha \right) \]

\[ \alpha \sim \frac{\Lambda}{\sqrt{n}} \]

Transport and phase diagram

Expect relativistic plasma physics of interacting particles and holes!
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Hydrodynamics?  
$\omega \ll T$

Interaction limited transport

Disorder limited transport

$T(\text{K})$

$10^{12} \text{m/s}$
Conductivity in and across the relativistic regime?

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+ Magnetotransport? e.g., Hall, Nernst effect?

\[ \sigma (e^2/h) \]

\[ V_0 (V) \]

\[ T(K) \]

\[ 10^{12}/m^2 \]

Other relativistic fluids:

- Bismuth (3d Dirac fermions with very small mass)
- Effective theories close to quantum phase transitions
- Conformal field theories
  E.g.: strongly coupled Non-Abelian gauge theories (QCD): tretament via AdS-CFT
Low energy effective theory at quantum phase transitions

Relativistic effective field theories $\leftrightarrow z = 1$; arise often due to particle-hole symmetry

Example: Superconductor-insulator transition (SIT)

Bhaseen, Green, Sondhi (PRL ’07).
Hartnoll, Kovtun, MM, Sachdev (PRB ’07)
SI-transition: Bose Hubbard model

Bose-Hubbard model

\[ H = -t \sum \langle ij \rangle b_j^+ b_i + U \sum n_i^2 - \mu \sum n_i \]

Coupling

\[ g \equiv \frac{t}{U} \] tunes the SI-transition

Diagram showing the Bose-Hubbard model's phase diagram with the quantum critical point (QCP) and the transition between commensurate Mott insulator and superfluid phases.
SI-transition: Bose Hubbard model

Bose-Hubbard model

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Coupling

\[ g \equiv \frac{t}{U} \] tunes the SI-transition

Effective action around \( g_c \) (\( \mu = 0 \)):

\[ S = \int d^2rd\tau \left[ | \partial_\tau \psi |^2 + v^2 \left| \vec{\nabla} \psi \right|^2 - g |\psi|^2 + \frac{u}{2} |\psi|^4 \right] \]
SI-transition: Bose Hubbard model

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\( \rightarrow \) Relativistic field theory in d=2+1
Questions

• **Transport characteristics** of the relativistic plasma in lightly doped graphene and close to quantum criticality?
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- How does the **relativistic regime** connect to **Fermi liquid** behavior at large doping?
Questions

• Transport characteristics of the relativistic plasma in lightly doped graphene and close to quantum criticality?

• How does the relativistic regime connect to Fermi liquid behavior at large doping?

• What is the range of validity of relativistic magneto-hydrodynamics?

• Beyond hydrodynamics?
Model of graphene

Graphene with Coulomb interactions and disorder

\[ H = H_0 + H_1 + H_{\text{dis}} \]

Tight binding kinetic energy

\[
H_0 = \sum_{a=1}^{N} \int dx \left[ \Psi_a^\dagger \left( i v_F \vec{\sigma} \cdot \vec{\nabla} + \mu \right) \Psi_a \right]
\]

\[
H_0 = \sum_{\lambda=\pm} \sum_{a=1}^{N} \int \frac{d^2 k}{(2\pi)^2} \lambda v_F k \gamma_{\lambda a}^\dagger (\vec{k}) \gamma_{\lambda a} (\vec{k})
\]
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B-field:

\[ i\vec{\nabla} \rightarrow i\vec{\nabla} - e\vec{A}/c \]

\[
H_0 = -\sum_{a=1}^{N} \int dx \left[ \Psi_a^\dagger \left( iv_F \vec{\sigma} \cdot \vec{\nabla} + \mu \right) \Psi_a \right]
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Coulomb interactions

\( H_1 = \frac{1}{2} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Psi_{a}^\dagger(k_2 - q) \Psi_{a}(k_2) V(q) \Psi_{b}^\dagger(k_1 + q) \Psi_{b}(k_1) \)

\( V(q) = \frac{2\pi e^2}{\varepsilon |q|} \)
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Coulomb marginally irrelevant!

\[ V (q) = \frac{2\pi e^2}{\varepsilon |q|} \]

\[ \alpha = \frac{e^2}{\varepsilon \hbar v_F} = O(1) \]

RG:

\[ \frac{d\alpha}{dl} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3) \]

\[ \alpha(T) = \frac{\alpha^0}{1 + (\alpha^0/4) \ln(\Lambda/T)} \sim \frac{4}{\ln(\Lambda/T)} \]
Model of graphene

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\[ H = H_0 + H_1 + H_{\text{dis}} \]

Tight binding kinetic energy

\[ H_0 = -\sum_{a=1}^{N} \int dx \left[ \bar{\Psi}_a \left( i v_F \vec{\sigma} \cdot \vec{\nabla} + \mu \right) \Psi_a \right] \]

\[ H_0 = \sum_{\lambda=\pm} \sum_{a=1}^{N} \int \frac{d^2 k}{(2\pi)^2} \lambda v_F k \gamma_{\lambda a}^\dagger(k) \gamma_{\lambda a}(k) \]

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Screening neglected (down by factor $\alpha$)

B-field:

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Coulomb marginally irrelevant!

\[ V(q) = \frac{2\pi e^2}{\varepsilon |q|} \]

Screening neglected (down by factor \( \alpha \))

Disorder: charged impurities

\[ H_{\text{dis}} = \int dx V_{\text{dis}}(x) \Psi_a^\dagger(x) \Psi_a(x) \]

\[ V_{\text{dis}}(x) = \sum_i \delta(x - x_i) \frac{Ze^2}{\varepsilon |x - x_i|} \]
Time scales

1. Inelastic scattering rate
   (Electron-electron interactions)

   \[ \tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \frac{1}{\max[1, \mu/T]} \]

Relativistic regime ($\mu < T$):
Relaxation rate set by temperature,
like in quantum critical systems!
Time scales


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   Relativistic regime (\( \mu < T \)):
   Relaxation rate set by temperature,
   like in quantum critical systems!

2. Elastic scattering rate
   (Scattering from charged impurities)

   \[ \tau_{\text{imp}}^{-1} \sim \frac{(Z e^2/\varepsilon)^2 \rho_{\text{imp}}}{\hbar} \frac{1}{\max[T, \mu]} \]

   Subdominant at high \( T \)
Time scales


1. Inelastic scattering rate
   (Electron-electron interactions)
   \[
   \tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \frac{1}{\max[1, \mu/T]}
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   (Scattering from charged impurities)
   Subdominant at high T
   \[
   \tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar}
   \]
   \[
   \tau_{imp}^{-1} \sim \left(\frac{Ze^2}{\epsilon}\right)^2 \frac{1}{\rho_{imp}} \frac{1}{\hbar} \max[T, \mu]
   \]

3. Deflection rate due to magnetic field
   (Cyclotron frequency of non-interacting
   particles with typical thermal energy)
   \[
   \tau_B^{-1} \sim \omega_c^{\text{typ}} \sim \frac{eBv_F^2}{\max[T, \mu]}
   \]
Regimes


1. Hydrodynamic regime: (collision-dominated)

\[ \tau_{ee}^{-1} \gg \tau_{imp}^{-1}, \tau_{B}^{-1}, \Omega \]
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(collision-dominated)

\[
\tau_{ee}^{-1} \gg \tau_{imp}^{-1}, \tau_B^{-1}, \omega
\]

2. Ballistic magnetotransport
(large field limit)

\[
\tau_B^{-1} > \tau_{ee}^{-1} >> \tau_{imp}^{-1}, \omega
\]

Regimes

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1. Hydrodynamic regime: (collision-dominated)

\[ \tau_{\text{ee}}^{-1} \gg \tau_{\text{imp}}^{-1}, \tau_{B}^{-1}, \omega \]

2. Ballistic magnetotransport (large field limit)

\[ \tau_{B}^{-1} > \tau_{\text{ee}}^{-1} \gg \tau_{\text{imp}}^{-1}, \omega \]

3. Disorder limited transport (inelastic scattering ineffective due to nearly conserved momentum)

\[ \mu \gg T \]

\[ \tau_{\text{ee}}^{-1} \geq \tau_{\text{imp}}^{-1} \]
Hydrodynamic Approach
Hydrodynamics

Hydrodynamic collision-dominated regime

Long times, Large scales

\[ t \gg \tau_{ee} \]

\[ \tau_{ee}^{-1} \gg \tau_{\text{imp}}^{-1}, \tau_{B}^{-1}, \omega \]
Hydrodynamics

Hydrodynamic collision-dominated regime

- Local equilibrium established: $T_{loc}(r), \mu_{loc}(r); \vec{u}_{loc}(r)$
- Study relaxation towards global equilibrium
- Slow modes: Diffusion of the density of conserved quantities:
  - Charge
  - Momentum
  - Energy

Long times, Large scales

$t \gg \tau_{ee}$

$\tau_{ee}^{-1} \gg \tau_{imp}^{-1}, \tau_{B}^{-1}, \omega$
Relativistic Hydrodynamics


Energy-momentum tensor

\[ T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu} \]

Current 3-vector

\[ J^\mu = \rho u^\mu + \nu^\mu \]

- \( u^\mu \): Energy velocity: \( u^\mu = (1,0,0) \rightarrow \) No energy current
- \( \nu^\mu \): Dissipative current ("heat current")
- \( \tau^{\mu\nu} \): Viscous stress tensor (Reynold’s tensor)
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+ Thermodynamic relations

\[ \varepsilon + P = Ts + \mu \rho, \quad d\varepsilon = T ds + \mu d\rho, \]
Relativistic Hydrodynamics


\[ J^\mu = \rho u^\mu + \nu^\mu \]

\[ T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu} \]

Conservation laws (equations of motion):

\[ \partial_\mu J^\mu = 0 \]

Charge conservation
Relativistic Hydrodynamics


\[ J^\mu = \rho u^\mu + v^\mu \]

\[ T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu} \]

Conservation laws (equations of motion):

\[ \partial_\mu J^\mu = 0 \quad \text{Charge conservation} \]

Energy/momentum conservation

\[ \partial_\nu T^{\mu\nu} = F^{\mu\nu}J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu} \delta^{\mu 0} \]

\[ F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix} \]

\[ \vec{E} = -i\frac{2\pi}{|k|} \rho_k \quad \text{Coulomb interaction} \]
Relativistic Hydrodynamics


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Coulomb interaction

Heat current and viscous tensor?
Relativistic Hydrodynamics


Heat current and viscous tensor?

Heat current \( Q^\mu = (\varepsilon + P)u^\mu - \mu J^\mu \)

\( \rightarrow \) Entropy current \( S^\mu = Q^\mu / T \)

Landau-Lifschitz, Relat. plasma physics
**Heat current and viscous tensor?**

Heat current \( Q^\mu = (\varepsilon + P)u^\mu - \mu J^\mu \)

→ Entropy current \( S^\mu = Q^\mu / T \)

\[ \partial_\mu S^\mu \equiv A_\alpha \left( \partial T, \partial \mu, F^{\mu \nu} \right) \nu^\alpha + B_\alpha \beta \left( \partial T, \partial \mu, F^{\mu \nu} \right) \tau^{\alpha \beta} \geq 0 \]

\[ \Rightarrow \nu^\mu = \text{const.} \times A^\mu \left( \partial T, \partial \mu, \partial u; F^{\mu \nu} \right) \]

\[ \tau^{\mu \nu} = \text{const.} \times B^{\mu \nu} + \text{const.} \times \delta^{\mu \nu} B^\alpha_\alpha \]
Relativistic Hydrodynamics

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Positivity of entropy production (Second law):

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\[ \nu^\mu = \sigma Q \left( g^{\mu\nu} + u^\mu u^\nu \right) \left[ - \partial_\nu \mu + F_\nu^{\lambda \mu} u^\lambda \right] + \mu \frac{\partial T}{T} \]

\[ \tau^{\mu\nu} = - (g^{\mu\lambda} + u^\mu u^\lambda) \left[ \eta \left( \partial_\lambda u^\nu + \partial^\nu u_\lambda \right) + (\zeta - \eta) \delta^\nu_\lambda \partial_\sigma u^\sigma \right] \]
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\[ B \text{ small!} \]
Relativistic Hydrodynamics


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Irrelevant for response at \( k \to 0 \)
Relativistic Hydrodynamics


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Irrelevant for response at \( k \to 0 \)

One single transport coefficient (instead of two)!
Meaning of $\sigma_Q$?

• Dimension of electrical conductivity

• At zero doping (particle-hole symmetry):

$$\sigma_Q = \sigma_{xx}(\rho_{\text{imp}} = 0)$$

= Universal d.c. conductivity of the pure system

Why is $\sigma_{xx}(\rho_{\text{imp}} = 0)$ finite??
Universal conductivity $\sigma_Q$

Standard situation: No particle-hole symmetry ($\rho \neq 0$)

- Current is carried predominantly by majority carriers
- Finite current implies finite momentum:

\[
\vec{J} \neq 0, \quad \vec{J}^E \neq 0
\]

- In the absence of impurities:
  Momentum conservation implies infinite conductivity!
Universal conductivity $\sigma_Q$


Particle-hole symmetry ($\rho = 0$)

- Key: Charge current without momentum (energy current)!

$\vec{J} \neq 0$, $\vec{P} = 0$

- Finite “quantum critical” conductivity!

Pair creation/annihilation leads to current decay

(particle) \quad \rightarrow \quad (hole)

\[ \vec{J} \neq 0, \quad \vec{P} = 0 \]
Universal conductivity $\sigma_0$


Quantum critical situation: Particle-hole symmetry ($\rho = 0$)

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- Finite “quantum critical” conductivity!
- As in quantum criticality:
  Relaxation time set by temperature alone

\[
\tau_{ee} \approx \frac{\hbar}{\alpha^2 k_B T}
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(pair) \rightarrow (hole)
Universal conductivity $\sigma_Q$


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\[ \sigma_{\text{Drude}} = \frac{e}{m} \rho \tau \rightarrow \sigma_Q \sim \frac{e}{k_B T / \nu^2} \left( e \frac{(k_B T)^2}{(\hbar \nu)^2} \right) \frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2} \frac{e^2}{\hbar} \]

→ Universal conductivity
Universal conductivity $\sigma_Q$


Quantum critical situation: Particle-hole symmetry ($\rho = 0$)

- Key: Charge current without momentum (energy current)
- Finite “quantum critical” conductivity!
- As in quantum criticality:
  Relaxation time set by temperature alone

\[ \tilde{J} \neq 0, \quad \tilde{P} = 0 \]

\[ \tau_{ee} \approx \frac{\hbar}{\alpha^2 k_B T} \]

$\rightarrow$ Universal conductivity

Exact (Boltzmann)

\[ \sigma_Q(\mu = 0) = \frac{0.76}{\alpha^2 k_B T} \]
Universal conductivity \( \sigma_Q \)

\[ K. \text{ Damle, S. Sachdev, (1996).} \]

Quantum critical situation: Particle-hole symmetry \((\rho = 0)\)

- Key: Charge current without momentum (energy current)

\[ (\text{particle}) \quad \text{ } \quad (\text{hole}) \]

\( \vec{J} \neq 0, \quad \vec{P} = 0 \)

- Finite “quantum critical” conductivity!
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\[\tau_{ee} \approx \frac{\hbar}{\alpha^2 k_B T}\]

\[\sigma_{\text{Drude}} = \frac{e}{m} \rho \tau \rightarrow \sigma_Q \sim \frac{e}{k_B T/\sqrt{2}} \left( \frac{e (k_B T)^2}{(\hbar \nu)^2} \right) \frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2} \frac{e^2}{h}\]

\[\sigma_Q(\mu = 0) = \frac{0.76}{\alpha^2} \frac{e^2}{h}\]

Marginal irrelevance of Coulomb:

\[\alpha \approx \frac{4}{\log(\Lambda/T)}\]
Thermoelectric response


Charge and heat current:

\[ J^\mu = \rho u^\mu - \nu^\mu \]
\[ Q^\mu = (\varepsilon + P) u^\mu - \mu J^\mu \]

Thermo-electric response

\[
\begin{pmatrix}
\hat{J} \\
\hat{Q}
\end{pmatrix} =
\begin{pmatrix}
\hat{\sigma} & \hat{\alpha} \\
T\hat{\alpha} & \hat{\kappa}
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
-\nabla T
\end{pmatrix}
\]

\[ \hat{\sigma} =
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
-\sigma_{xy} & \sigma_{xx}
\end{pmatrix}
\]

etc.
Thermoelectric response


Charge and heat current:

\[ J^\mu = \rho u^\mu - \nu^\mu \]
\[ Q^\mu = (\varepsilon + P)u^\mu - \mu J^\mu \]

Thermo-electric response

\[
\begin{pmatrix}
\hat{J} \\
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\end{pmatrix} =
\begin{pmatrix}
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T\hat{\alpha} & \frac{\hat{\kappa}}{\kappa}
\end{pmatrix}
\begin{pmatrix}
\hat{E} \\
-\nabla T
\end{pmatrix}
\]
\[ \hat{\sigma} =
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
-\sigma_{xy} & \sigma_{xx}
\end{pmatrix}
\]

etc.

i) Solve linearized hydrodynamic equations
ii) Read off the response functions (Kadanoff & Martin 1960)
Results from Hydrodynamics
Response functions at B=0

Symmetry $z \rightarrow -z : \quad \sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

**Longitudinal conductivity:**

$$\sigma_{xx}(\omega, k; B = 0) = \left( \sigma_Q + \frac{\rho^2 \tau}{P + \epsilon_1 - i\omega\tau} \right)$$

Universal conductivity at the quantum critical point $\rho = 0$

Drude-like conductivity, divergent for

Momentum conservation ($\rho \neq 0$)! $\tau \rightarrow \infty, \omega \rightarrow 0, \rho \neq 0$
Response functions at B=0

Symmetry $z \rightarrow -z$ : $\sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B = 0) = \left(\sigma_Q + \frac{\rho^2}{P + \varepsilon 1 - i\omega \tau}\right) \left[1 - i\frac{g k}{\omega}\left(\sigma_Q + \frac{\tau}{1 - i\omega \tau} \frac{\rho^2}{P + \varepsilon}\right)\right] + \mathcal{O}(k^2)$$

Coulomb correction ($g = 2\pi e^2$)
Response functions at $B=0$

Symmetry $z \rightarrow -z : \quad \sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

**Longitudinal conductivity:**

$$\sigma_{xx}(\omega, k; B = 0) = \left( \sigma_Q + \frac{\rho^2}{P + \varepsilon(1 - i\omega\tau)} \right) \left[ 1 - \frac{i g k}{\omega} \left( \sigma_Q + \frac{\tau}{1 - i\omega\tau} \frac{\rho^2}{P + \varepsilon} \right) \right] + O(k^2)$$

**Thermal conductivity:**

$$\kappa_{xx}(\omega, k; B = 0) = \sigma_Q \frac{\mu^2}{T} + \frac{s^2 \tau}{P + \varepsilon(1 - i\omega\tau)} + O(k^2).$$

Relativistic Wiedemann-Franz-like relations between $\sigma$ and $\kappa$ in the quantum critical window!
Response functions at B=0

Symmetry $z \rightarrow -z$ : $\sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B = 0) = \left(\sigma_Q + \frac{\rho^2}{P + \varepsilon 1 - i\omega\tau}\right) \left[1 - \frac{igk}{\omega} \left(\sigma_Q + \frac{\tau}{1 - i\omega\tau} \frac{\rho^2}{P + \varepsilon}\right)\right] + O(k^2)$$

Coulomb correction $(g = 2\pi e^2)$

Thermopower:

$$\alpha_{xx}(\mu, \omega = 0) = -\frac{\pi^2}{3e} k_B^2 T \frac{d\sigma(\mu, \omega = 0)}{d\mu}$$

Relativistic fluid!

Only valid in the Fermi liquid regime, but violated in the relativistic window.
B > 0: Cyclotron resonance

E.g.: Longitudinal conductivity

\[ \sigma_{xx}(\omega) = \sigma_Q \frac{\omega \left( \omega + i\gamma + i\omega_c^2/\gamma \right)}{\left( \omega + i\gamma \right)^2 - \omega_c^2} \]

Poles in the response

\[ \omega = \pm \omega_c^{QC} - i\gamma - i/\tau \]

Collective cyclotron frequency of the relativistic plasma

\[ \omega_c^{QC} = \frac{\rho B}{(\epsilon + P)/v_F^2} \leftrightarrow \omega_c^{FL} = \frac{e B}{m} \]
B > 0 : Cyclotron resonance

E.g.: Longitudinal conductivity

\[ \sigma_{xx}(\omega) = \sigma_Q \frac{\omega (\omega + i\gamma + i\omega_c^2 / \gamma)}{(\omega + i\gamma)^2 - \omega_c^2} \]

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Collective cyclotron frequency of the relativistic plasma

\[ \omega_c^{QC} = \frac{\rho B}{(\varepsilon + P)/v_F^2} \quad \leftrightarrow \quad \omega_c^{FL} = \frac{e B}{m} \]

Intrinsic, interaction-induced broadening

(\leftrightarrow Galilean invariant systems:
No broadening due to Kohn’s theorem)

\[ \gamma = \sigma_Q \frac{B^2}{(\varepsilon + P)/v_F^2} \]
B > 0 : Cyclotron resonance

Longitudinal conductivity

$$\sigma_{xx}(\omega, k) = \sigma_Q \frac{(\omega + i/\tau) (\omega + i/\tau + i\gamma + i\omega_c^2/\gamma)}{(\omega + i/\tau + i\gamma)^2 - \omega_c^2}$$

Poles in the response

$$\omega = \pm \omega_Q - i\gamma - i/\tau$$
Can the resonance be observed?

\[ \omega = \pm \omega_c - i\gamma - i/\tau \]

\[ v_F = 1.1 \cdot 10^6 \text{ m/s} \approx c/300 \]

Conditions to observe collective cyclotron resonance

- Collison-dominated regime
  \[ \hbar \omega_c \ll \alpha^2 k_B T \]

- Small broadening
  \[ \gamma, \tau^{-1} < \omega_c \]

- Quantum critical regime
  \[ \rho \leq \rho_{th} = \frac{(k_B T)^2}{(h v_F)^2} \]

- High T: no Landau quantization
  \[ E_{LL} = h \nu_F \sqrt{\frac{2eB}{hc}} \ll k_B T \]

Parameters:

- \( T \approx 300K \)
- \( B \approx 0.1T \)
- \( \rho \approx 10^{11} \text{ cm}^{-2} \)
- \( \omega_c \approx 10^{13} \text{ s}^{-1} \)
Does relativistic hydrodynamics apply?

- Do $T$ and $\mu$ not break relativistic invariance?
- Validity at large chemical potential?
- Beyond linearization in magnetic field?
- Treatment of disorder?
**Boltzmann Approach**

*MM, L. Fritz, and S. Sachdev, cond-mat 0805.1413.*

→ Recover and refine the hydrodynamic description

→ Describe relativistic-to-Fermi-liquid crossover

→ Go beyond hydrodynamics
\( \sigma_Q \text{ from Boltzmann} \)

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

Boltzmann equation in Born approximation

\[
\left( \partial_t + e[E + v \times B] \cdot \frac{\partial}{\partial k} \right) f_{\pm}(k, t) = \alpha^2 I_{\text{coll}}^{\text{Cb}}[k, t \mid \{f_{\pm}(k', t)\}] + \Delta I_{\text{coll}}^{\text{dis}}[k, t \mid \{f_{\pm}(k', t)\}]
\]

\[
\begin{array}{lllllll}
\text{a.)} & +, i & +, i & +, i & +, i & +, i & +, i \\
& \quad & & \quad & & \quad & \\
& -, i & -, i & -, i & -, i & -, j & -, j \\
\end{array}
\]

\[
\begin{array}{lllllll}
\text{b.)} & +, i & +, i & +, i & +, i & +, i \\
& \quad & & \quad & & \quad & \\
& +, i & +, i & +, i & +, j & +, j \\
\end{array}
\]
\( \sigma_Q \) from Boltzmann

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

Boltzmann equation in Born approximation

\[
\left( \frac{\partial}{\partial t} + e [E + v \times B] \cdot \frac{\partial}{\partial k} \right) f_\pm (k, t) = \alpha^2 I^{Cb}_{\text{coll}} [k, t \mid \{ f_\pm (k', t) \}] + \Delta I^{\text{dis}}_{\text{coll}} [k, t \mid \{ f_\pm (k', t) \}]
\]

Linearization:

\[
f_\pm (k, t) = f_{\pm}^{eq} (k, t) + \delta f_\pm (k, t)
\]
\[ \sigma_Q \text{ from Boltzmann} \]

*L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289*

Boltzmann equation in Born approximation

\[
\left( \partial_t + e [E + v \times B] \cdot \nabla \right) f_\pm (k, t) = \alpha^2 I_{\text{coll}}^\text{Cb} [k, t | \{ f_\pm (k', t) \}] + \Delta I_{\text{coll}}^\text{dis} [k, t | \{ f_\pm (k', t) \}]
\]

Linearization:

\[
f_\pm (k, t) = f_{\pm}^{eq} (k, t) + \delta f_\pm (k, t)
\]

Great simplification: Divergence of forward scattering amplitude in 2d

At p-h symmetry:

\[
f_\pm (k, t) = f_{\pm}^{eq} (k, \mu \rightarrow \mu_{eq} + \delta \mu (t)) ; \quad \delta \mu = C(t) \frac{E \cdot k}{k}
\]

→ Equilibration along unidimensional spatial directions
\( \sigma_Q \) from Boltzmann

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

Boltzmann equation in Born approximation

\[
\left( \partial_t + e [E + v \times B] \cdot \frac{\partial}{\partial k} \right) f_\pm (k, t) = \alpha^2 I_{\text{coll}}^{Cb} [k, t \mid \{ f_\pm (k', t) \}] + \Delta I_{\text{coll}}^{\text{dis}} [k, t \mid \{ f_\pm (k', t) \}]
\]

Linearization:

\[
f_\pm (k, t) = f_\pm^{eq} (k, t) + \delta f_\pm (k, t)
\]

Great simplification: Divergence of forward scattering amplitude in 2d

\[
\text{Amp} \left[ \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \right] \rightarrow \infty
\]

→ Equilibration along unidimensional spatial directions

At p-h symmetry:

\[
f_\pm (k, t) = f_\pm^{eq} (k, \mu \rightarrow \mu_{eq} + \delta \mu (t)) \quad \delta \mu = C(t) \frac{E \cdot k}{\hbar}
\]

\[
\sigma_Q (\mu = 0) \approx \frac{0.76 \, e^2}{\alpha^2 \, \hbar}
\]
σ₀ from Boltzmann

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

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\[
\left( \partial_t + e [E + v \times B] \cdot \frac{\partial}{\partial k} \right) f_\pm (k, t) = \alpha^2 I^{Cb}_{\text{coll}} [k, t \mid \{f_\pm (k', t)\}] + \Delta I^{\text{dis}}_{\text{coll}} [k, t \mid \{f_\pm (k', t)\}]
\]

General analysis in linear response:

\[
f_\lambda (r, k, \omega) = 2\pi \delta (\omega) f^0_\lambda (k, T(r)) + f^0_{\lambda k} \left[ 1 - f^0_{\lambda k} \right] \frac{v_F}{T} e_k \cdot \left[ eE(\omega) g^{(E)}_{\parallel,\lambda} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g^{(T)}_{\parallel,\lambda} \left( \frac{v_F k}{T}, \omega \right) \right]
\]

\[
+ f^0_{\lambda k} \left[ 1 - f^0_{\lambda k} \right] \frac{v_F}{T^2} (e_k \times e_z) \cdot \left[ E(\omega) g^{(E)}_{\perp,\lambda} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g^{(T)}_{\perp,\lambda} \left( \frac{v_F k}{T}, \omega \right) \right]
\]
σ_Q from Boltzmann

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

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\left( \partial_t + e [E + v \times B] \cdot \frac{\partial}{\partial k} \right) f_{\pm}(k, t) = \alpha^2 I_{\text{coll}}^{cb}[k, t | \{ f_{\pm}(k', t) \}] + \Delta I_{\text{coll}}^{\text{dis}}[k, t | \{ f_{\pm}(k', t) \}]
\]

General analysis in linear response:

\[
f_{\lambda}(r, k, \omega) = 2\pi \delta(\omega) f_{\lambda}^0(k, T(r)) + f_{\lambda k}^0 \left[ 1 - f_{\lambda k}^0 \right] \frac{v_F}{T^2} e_k \cdot \left[ e E(\omega) g_{\parallel,\lambda}^{(E)} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g_{\parallel,\lambda}^{(T)} \left( \frac{v_F k}{T}, \omega \right) \right] + f_{\lambda k}^0 \left[ 1 - f_{\lambda k}^0 \right] \frac{v_F}{T^2} (e_k \times e_z) \cdot \left[ e E(\omega) g_{\perp,\lambda}^{(E)} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g_{\perp,\lambda}^{(T)} \left( \frac{v_F k}{T}, \omega \right) \right]
\]

Central element of analysis: Choose appropriate basis \( g_{\lambda=\pm}(k, t) = \sum_n a_n \phi_n(\lambda, k) \)

\[
\phi_0(\lambda, k) = k, \quad \text{Momentum or energy-current mode}
\]

\[
\phi_1(\lambda, k) = \lambda, \quad \text{Charge current mode}
\]
σ₀ from Boltzmann

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

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\]

\[
+ f^0_\lambda \left[1 - f^0_\lambda \right] \frac{v_F}{T^2} (e_k \times e_z) \cdot \left[ E(\omega) g^{(E)}_{\perp, \lambda} \left( \frac{v_F k}{T}, \omega \right) + \nabla T(\omega) g^{(T)}_{\perp, \lambda} \left( \frac{v_F k}{T}, \omega \right) \right]
\]

Central element of analysis: Choose appropriate basis \( g_{\lambda=\pm}(k, t) = \sum_n a_n \phi_n(\lambda, k) \)

\( \phi_0(\lambda, k) = k \),  Momentum or energy-current mode

\( \phi_1(\lambda, k) = \lambda \),  Charge current mode

\[
\sum_\lambda \int d^2k f^0_{\lambda k} (1 - f^0_{\lambda k}) \phi_{n \geq 2}(\lambda, k) \phi_{0, 1}(\lambda, k) = 0
\]

Relativistic dispersion ensures that \( \phi_0 \) only couples to \( \phi_1 \) for clean systems!
Conductivity: $\sigma_Q$

*L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289*

**General doping:**

**Clean system:**

$$\sigma_{xx}(\omega; \mu, \Delta = 0) = e^2 \frac{\rho^2 v_F^2}{\varepsilon + P} \frac{1}{(-i\omega)} + \sigma_Q. $$

**Precise expression for $\sigma_Q$:**

$$\sigma_Q(\mu, \omega) = \frac{e^2}{\hbar} \frac{1}{\alpha^2} \frac{2 \hat{g}_1}{N} \left[ I^{(1)}_+ - \frac{\rho^2 (\hbar v)^2}{(\varepsilon + P)T} \right]^2 \frac{1}{1 - i\omega \tau_{ee}}$$
Conductivity: $\sigma_Q$

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

General doping:

Clean system:

$$\sigma_{xx}(\omega; \mu, \Delta = 0) = e^2 \frac{\rho^2 v_F^2}{\varepsilon + P (-i\omega)} + \sigma_Q.$$ 

Precise expression for $\sigma_Q$!

Gradual disappearance of relativistic physics

Will appear in all Boltzmann formulae below!
Conductivity: crossover

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

General doping:

Lightly disordered system:

\[
\sigma_{xx}(\omega; \mu, \Delta) = \frac{e^2}{\tau_{\text{imp}}^{-1} - i\omega} \frac{\rho^2 v_F^2}{\epsilon + P} + \sigma_Q + \delta\sigma(\Delta, \omega, \mu)
\]

\[
\delta\sigma(\Delta, \omega, \mu) = \mathcal{O}(\Delta/\alpha^2)
\]

← Correction to hydrodynamics
Conductivity: crossover

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

General doping:

Lightly disordered system:

\[ \sigma_{xx}(\omega; \mu, \Delta) = \frac{e^2}{\tau_{\text{imp}}(1 - i\omega/\mu)} \rho^2 \frac{v_F^2}{\mu} + \sigma_Q + \delta\sigma(\Delta, \omega, \mu) \]

\[ \delta\sigma(\Delta, \omega, \mu) = \mathcal{O}(\Delta/\alpha^2) \]

Fermi liquid regime:

\[ \sigma_{xx}(\omega = 0; \mu \gg T) \approx \frac{e^2 \rho^2 v_F^2 \tau_{\text{imp}}}{\varepsilon + P} \]

\[ = \frac{2}{\pi} \frac{1}{(Z\alpha)^2} \frac{e^2}{h} \frac{\rho}{\rho_{\text{imp}}} \]

\[ \rho_{\text{imp}} = \rho \]

\[ \rho \]

\[ \rho_{\text{imp}} \]

← Correction to hydrodynamics
Conductivity: crossover

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= \frac{2}{\pi} \frac{1}{(Z\alpha)^2} \frac{e^2}{\hbar} \frac{\rho}{\rho_{\text{imp}}}
\]
Conductivity: crossover

L. Fritz, J. Schmalian, MM, and S. Sachdev, condmat 0802.4289

General doping:

Lightly disordered system:

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\sigma_{xx}(\omega; \mu, \Delta) = \frac{e^2}{\tau_{\text{imp}}^{-1} - i\omega} \left( \frac{\rho^2 v_F^2}{\varepsilon + P} \right) + \sigma_Q + \delta\sigma(\Delta, \omega, \mu)
\]

\[
\delta\sigma(\Delta, \omega, \mu) = \mathcal{O}(\Delta/\alpha^2)
\]

← Correction to hydrodynamics

Fermi liquid regime:

\[
\sigma_{xx}(\omega = 0; \mu \gg T) \approx \frac{e^2 \rho^2 v_F^2 \tau_{\text{imp}}}{\varepsilon + P}
\]

\[
= \frac{1}{\pi (Z\alpha)^2} \frac{e^2}{h} \frac{\rho}{\rho_{\text{imp}}}
\]


Impurity limited conductivity

Universal Cb limited conductivity
Magnetotransport

- **Strategy**: describe the slow dynamics of the momentum mode $\varphi_0$ in very weak disorder and moderate magnetic field
Magnetotransport

- **Strategy**: describe the slow dynamics of the momentum mode $\varphi_0$ in very weak disorder and moderate magnetic field

**Result**: Full thermoelectric response (for general $B$) obtained in terms of thermodynamic quantities + only 2 independent transport coefficients (collision matrix elements)!
Magnetotransport

- **Strategy**: describe the slow dynamics of the momentum mode $\varphi_0$ in very weak disorder and moderate magnetic field

**Result**: Full thermoelectric response (for general $B$) obtained in terms of thermodynamic quantities + only 2 independent transport coefficients (collision matrix elements)!

- At small $B$, one transport coefficient is subdominant
  $\rightarrow$ Relativistic hydrodynamics with only one transport coefficient $\sigma_Q$ is recovered!

\[
\tau_{ee}^{-1} \gg \tau_B^{-1}
\]

\[
\sigma_{xx}(\omega, B) = \sigma_{xx}^{\text{MHD}}(\omega, B) + \mathcal{O}(b/\alpha^2, \omega/\alpha^2)
\]

Corrections to hydrodynamics
Magnetotransport

• **Strategy:** describe the slow dynamics of the momentum mode $\varphi_0$ in very weak disorder and moderate magnetic field

**Result:** Full thermoelectric response (for general $B$) obtained in terms of thermodynamic quantities + only 2 independent transport coefficients (collision matrix elements)!

• At small $B$, one transport coefficient is subdominant
  $\rightarrow$ Relativistic hydrodynamics with only one transport coefficient $\sigma_Q$ is recovered!

Cyclotron resonance:

$\tau_{ee}^{-1} \gg \tau_B^{-1}$

Hydrodynamics
Boltzmann
Cyclotron resonance revisited

Crossover to Fermi liquid regime:

• Semiclassical $\omega_c$ recovered at $\mu \gg T$

• Broadening goes to zero - Kohn’s theorem recovered: Non-broadening of the resonance for a single parabolic band.

\[
\omega_c^{(0)} = \frac{\rho B}{\epsilon + P} \rightarrow \frac{eB}{\mu/v_F^2} = \frac{eB}{\hbar k_F/v_F}
\]

\[
\gamma \equiv \frac{\sigma_Q B^2 v_F^2}{(\epsilon + P)}
\]

\[
\gamma \propto \sigma_Q(\mu)^{\mu \gg T} \rightarrow 0
\]
Cyclotron resonance revisited

Beyond hydrodynamics: Towards ballistic magnetotransport

\[ \mu = T \]

Large fields

\[ \tau_B^{-1} > \tau_{ee}^{-1} >> \tau_{\text{imp}}^{-1}, \omega \]

**Resonance**

**Damping**
Strongly coupled liquids

Same trends as in exact (AdS-CFT) results for strongly coupled relativistic fluids!

S. Hartnoll, C. Herzog (2007)

Graphene

Resonance

Damping

\( \mathcal{N} = 4 \) SUSY SU(N) gauge theory [flows to CFT at low energy]
• Relativistic physics in graphene and quantum critical systems

• Hydrodynamic description:
  → collective cyclotron resonance in the relativistic regime
  → covariance: 6 frequency dependent response functions given by thermodynamics and *only one* parameter $\sigma_Q$.

• Boltzmann approach
  → confirmed and refined hydrodynamic description
  → understood relativistic-to-Fermi liquid crossover:
    • From universal Coulomb-limited to disorder-limited linear conductivity in graphene
    • From collective-broadened to semiclassic sharp cyclotron resonance
  → beyond hydrodynamics: describe large fields and disorder