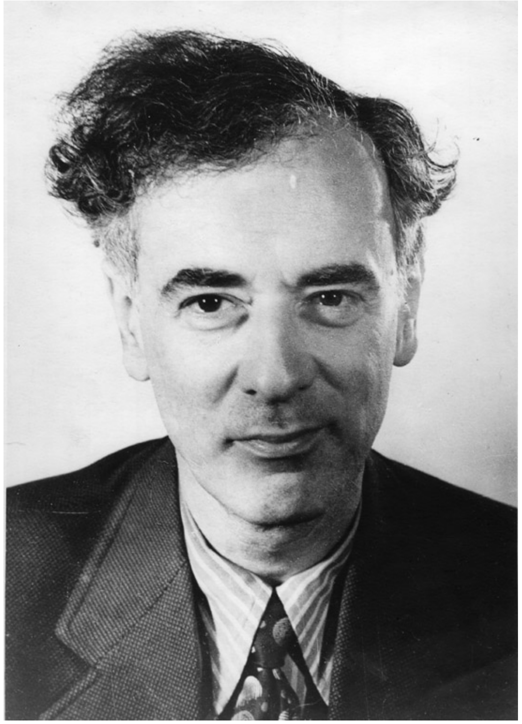
Weakly interacting Bose gas in disordered environment



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Outline

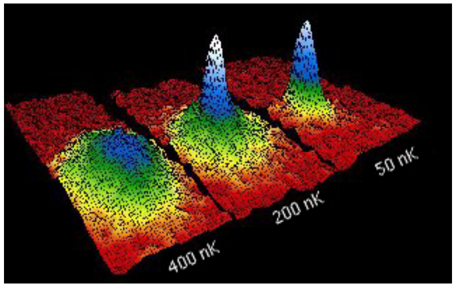
- Introduction
- Single-atom levels in a random potential
- Weakly repulsive gas in a random potential
- Weakly repulsive gas in a harmonic trap
- Conclusions

Purposes: Semiquantitative analysis of the phase states of the gas in a box
The size and shape of the atomic cloud in a harmonic trap

Introduction

Bose-Einstein-condensation: finite part of atoms in the state with minimal energy.

Examples: Superfluid ^{4}He, laser cooled atoms in a trap



Disorder: Superfluid He in a porous media: J. Reppy and coworkers, J. Low Temp. Phys. **87**, 205 (1992) and references therein: Breakdown of superfluidity at strong disorder

Bose-Einstein Condensate in a Random Potential

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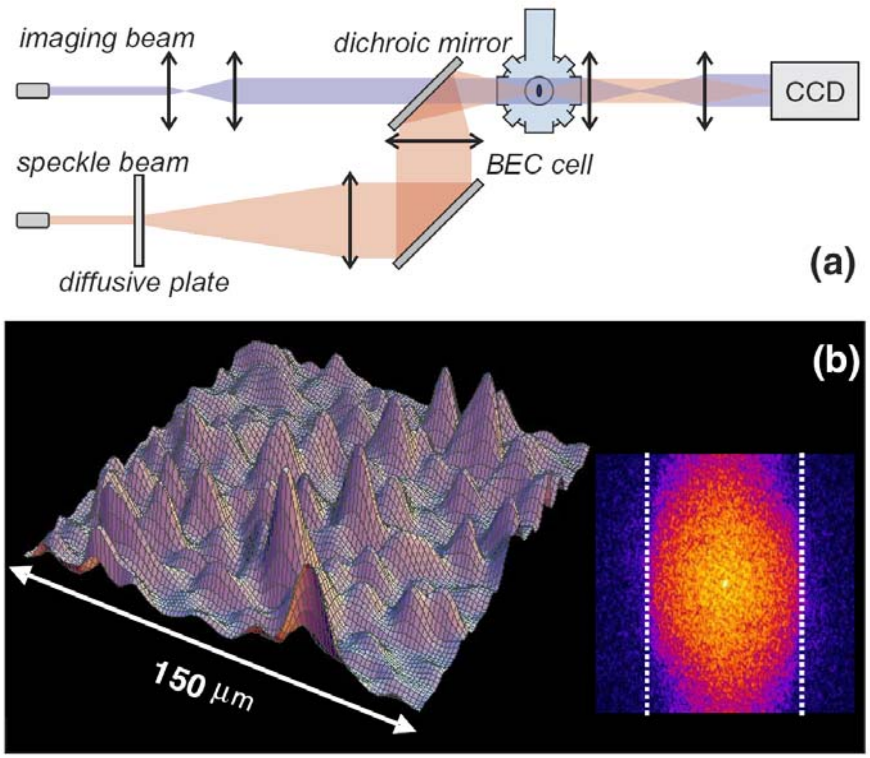


FIG. 1 (color online). (a) Optical setup for both the speckle potential and the imaging beam for the BEC. The axial direction of the magnetic trap is in the vertical direction of the figure. (b) 3D representation of the speckle potential (left) and its Fourier transform (right). The dotted lines correspond to a length scale of about 10 μm in the axial direction.

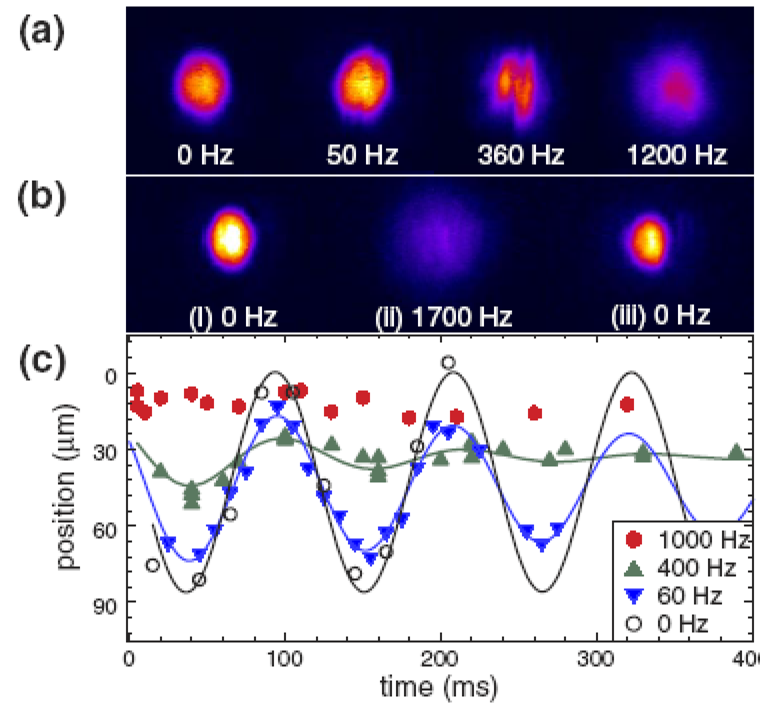


FIG. 2 (color online). (a) Density profile of the BEC after 28 ms of expansion from the combined magnetic and speckle potential for varying speckle potential intensities V_s (indicated in bottom part). To demonstrate that the transfer to the speckle is adiabatic (b) shows the density profiles with (i) no speckle potential, (ii) when the speckle potential is abruptly switched off, and (iii) when it is adiabatically ramped off. (c) Dipole oscillations in the combined magnetic and speckle potential for varying V_s.

Theoretical works:

Corrections to the Bogolyubov theory caused by a weak disorder

K. Huang and H.F. Meng, Phys. Rev. Lett. **69,** 644 (1992);
S. Giorgini, L.P. Pitaevsky and S. Stringari, Phys. Rev. B **49**, 12938 (1994)
A. V. Lopatin and V.M. Vinokur, Phys. Rev. Lett. **88**, 235503 (2002)

Possible Bose-glass state

M.P.A. Fisher et al., Phys. Rev. B **50**, 546 (1989)

R.T. Scalettar et al., Phys. Rev. Lett. **66,** 3144 (1991)
W. Krauth et al., Phys. Rev. Lett. **67**, 2307 (1991)

Spin model, transition from the normal state to superfluid

M. Ma, B.I. Halperin and P.A. Lee, Phys. Rev. B **34**, 3136 (1986)

Bose Hubbard model, scaling near transition point

M.P.A. Fisher et al., Phys. Rev. B **50**, 546 (1989)

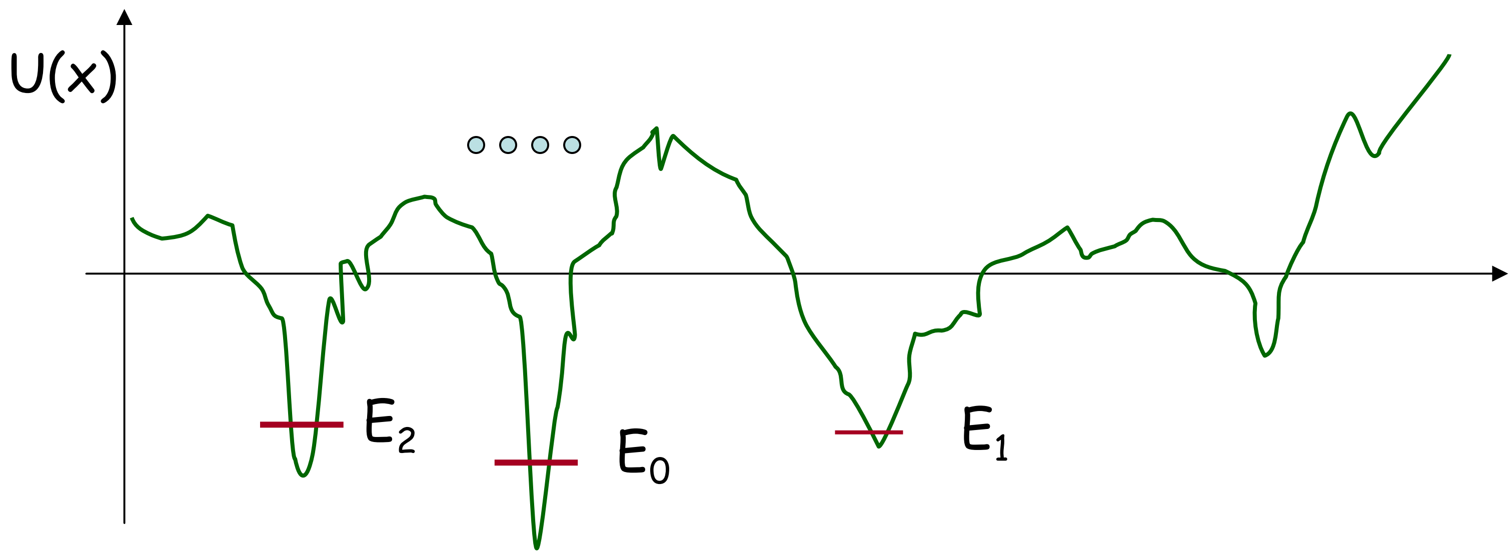
One-dimensional Hubbard model

T. Giamarchi and H. Schulz, Europhys. Lett. **3**, 1287 (1987)

Single atom levels in a random potential

Ideal Bose gas

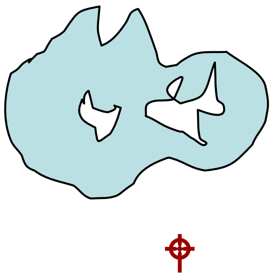
Ground state



Non-ergodic ground state or quenched distribution dependent on the cooling speed

Gaussian white noise

$$\left\langle U(\mathbf{x})U(\mathbf{x}')\right\rangle = \kappa^2 \delta(\mathbf{x}-\mathbf{x}')$$



Independent probability distribution in each point:

$$dW\left[U(\mathbf{x}),\Omega\right] = \exp\left[-\frac{1}{2\kappa^2}\int_{\Omega} U^2(\mathbf{x})\,d\Omega\right]\prod_{\mathbf{x}\in\Omega}\frac{\sqrt{\Delta\Omega}\,dU(\mathbf{x})}{\sqrt{2\pi\kappa}}$$

Energy levels in the random potential:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U(\mathbf{x})\psi = E[U]\psi$$

Density of states: $\nu(E,\Omega)=\int \delta\left(E-E[U]\right) dW\left[U(\mathbf{x}),\Omega\right]$

$$dW\left[U(\mathbf{x}),\Omega\right]=\exp\left[-\frac{1}{2\kappa^2}\int_{\Omega} U^2(\mathbf{x})\,d\Omega\right]\prod\nolimits_{\mathbf{x}\in\Omega}\frac{\sqrt{\Delta\Omega}\,dU(\mathbf{x})}{\sqrt{2\pi\kappa}}$$

Deep levels: $E<0;\ |E| \gg E_{\mathfrak{L}}=\dfrac{m^2\kappa^4}{\hbar^2}$

I.M. Lifshitz, Zittartz and Langer, Halperin and Lax, 1966

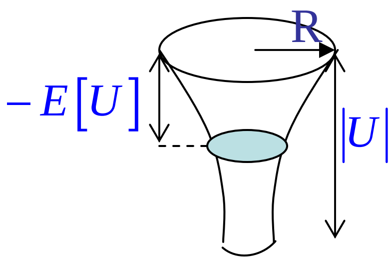
Minimize $\int_{\Omega} U^2(\mathbf{x})\,d\Omega$ at fixed $E[U]$

The optimal fluctuation potential well is spherically symmetric.
The optimal level is the only bound state in this well.

$U(\mathbf{x})=\lambda\left|\psi(\mathbf{x})\right|^2$ Ginzburg-Landau equation

Fluctuation potential well has the same linear size as the wave function

Semiquantitative approach



Typical potential well with a deep level

All calculations for 3d space

Balance of kinetic and potential energy: $|E| \sim |U| \sim \frac{\hbar^2}{2mR^2}$

Probability to find energy less than –*E* or part of space occupied by the potential wells with the size of the quantum state less than R:

$$P(R) = \exp\left(-\frac{U^2\Omega}{2\kappa^2}\right) = \exp\left(-\frac{\pi\hbar^4}{6m^2\kappa^2 R}\right)$$

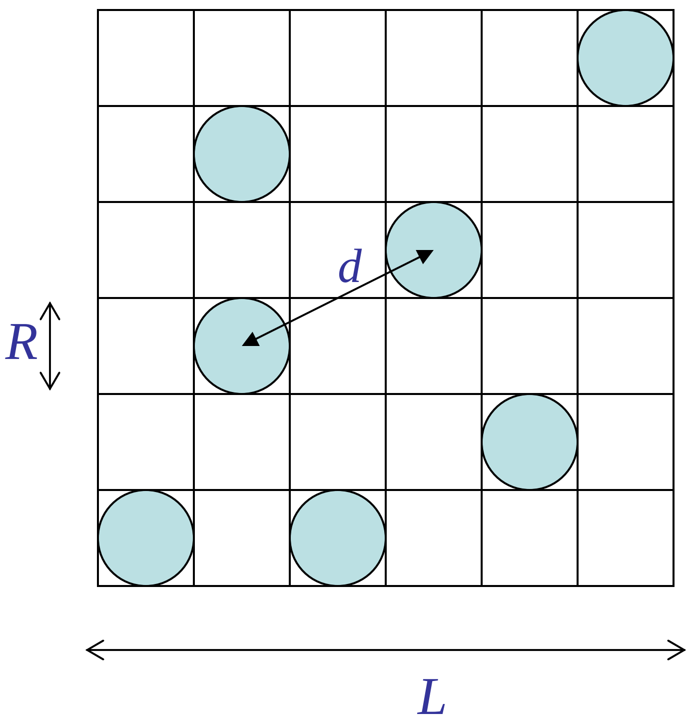
Explanation: $\int_\Omega U^2(\mathbf{x})\,d\Omega \approx U^2\Omega \qquad \Omega \approx \frac{4}{3}\pi R^3$

Larkin length: $\mathfrak{L} = \frac{\pi\hbar^4}{6m^2\kappa^2}$

Probability: $P(R) = \exp\left(-\frac{\mathfrak{L}}{R}\right)$

Preexponent: inessential

Valid at $R \gg \mathfrak{L}$ $\qquad P(\mathfrak{L}) \sim 1 \qquad E(\mathfrak{L}) \equiv E_{\mathfrak{L}} = \frac{\hbar^2}{2m\mathfrak{L}^2}$ Levels of Larkin size percolate



Density of cells with the quantum states of the size *R*

$$n_w(R) = R^{-3}\exp\left(-\frac{\mathcal{L}}{R}\right)$$

Average distance between the states of the size *R:*

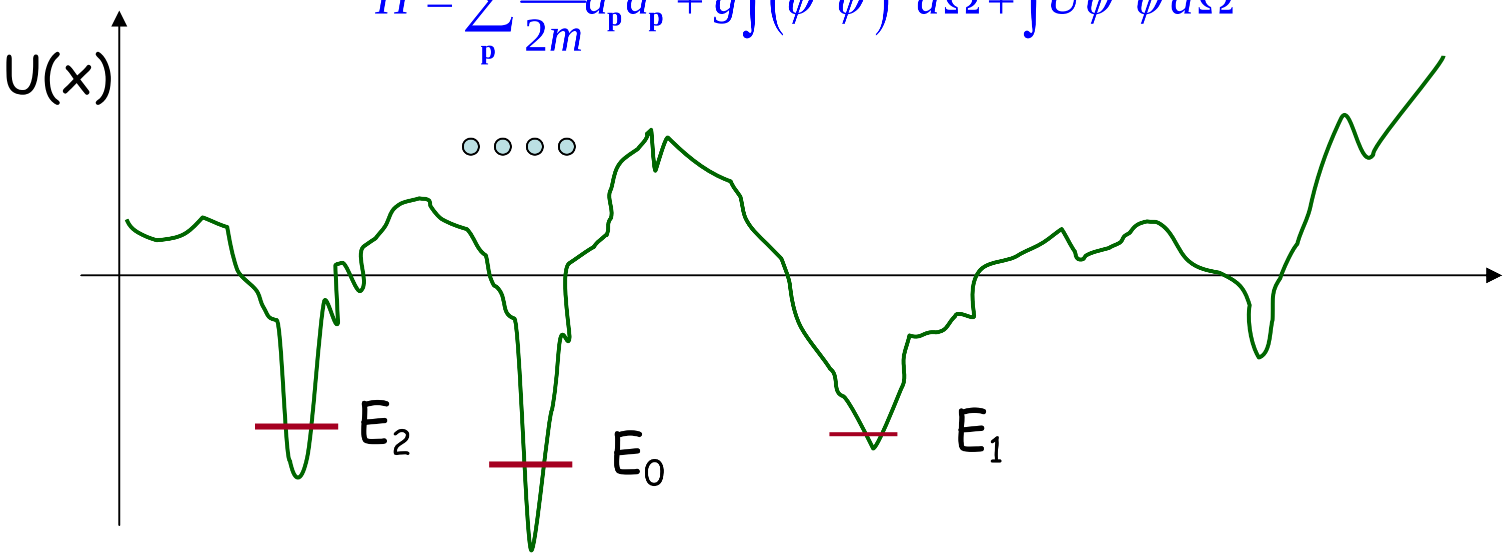
$$d(R) = \left[n_w(R)\right]^{-1/3} = R\exp\left(\frac{\mathcal{L}}{3R}\right)$$

Tunneling amplitude between neighboring states of the size *R*

$$t(R) = \exp\left(-\frac{\sqrt{2m|U|}\,d(R)}{\hbar}\right) = \exp\left[-\exp\left(\frac{\mathcal{L}}{3R}\right)\right]$$

Weakly repulsive gas in a random potential

$$H=\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}+g\int\left(\psi^{\dagger}\psi\right)^2 d\Omega+\int U\psi^{\dagger}\psi\, d\Omega$$



Due to repulsion atoms are distributed between different wells

Repulsion restores ergodicity

Repulsion energy per particle is $gn_p(R)$

$n_p(R)$ is the density of particles in the well of the size R

Let the potential wells with the radii up to R are filled. Then:

$$n_p(R) = \frac{n}{n_w(R)} \frac{3}{4\pi R^3} = \frac{3n}{4\pi} \exp\left(\frac{\mathcal{L}}{R}\right)$$

Number of particles per well

Energy per particle: $$\mu = gn_p(R) - \frac{\hbar^2}{2mR^2}$$

Energy of the random well

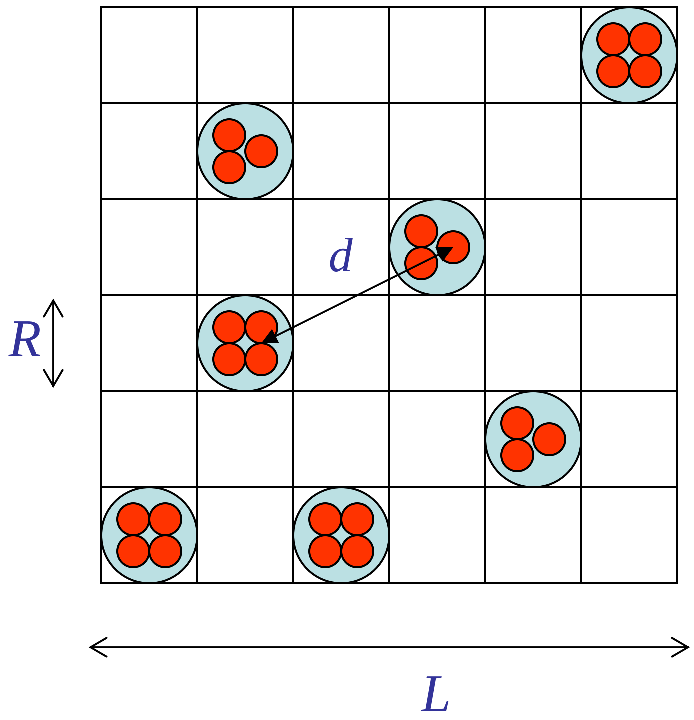
Mininization over R: $$R(n) = \frac{\mathcal{L}}{\ln \frac{n_c}{n}} \qquad n_c = \left(3\mathcal{L}^2 a\right)^{-1} \qquad g = \frac{4\pi\hbar^2 a}{m}$$

Typical radius reached

Critical density

Scattering length

Chemical potential: $$\mu = -\frac{\hbar^2}{2m\mathcal{L}^2}\left(\ln \frac{n_c}{n}\right)^2 = -\mathcal{E}_{\mathcal{L}}\left(\ln \frac{n_c}{n}\right)^2$$



Maximal size of the well occupied by atoms:

$$R(n) = \mathcal{L} \ln \frac{n_c}{n} \qquad n_c = \left(3\mathcal{L}^2 a\right)^{-1}$$

Distance between occupied wells:

$$r(n) = \mathcal{L} \left(\frac{n_c}{n}\right)^{1/3} \ln \frac{n_c}{n} \gg R(n)$$

Tunneling amplitude between wells:

$$t(n) = \exp\left[-\left(\frac{n_c}{n}\right)^{1/3}\right]$$

Number of particles per well: $\mathcal{N} = \dfrac{\mathcal{L}}{3a \ln\left(n_c / n\right)}$

Partial conclusions

At n much less than n_c different wells are exponentially weakly coupled and have well defined number of particles. Therefore the phase in each well is uncertain. Random singlet. No coherence, no superfluidity, Bose glass?

At n approaching n_c distances between the wells become of the same order of magnitude as their radius, the overlapping factor t becomes of the order of 1. Strong growth of the correlation between phases.

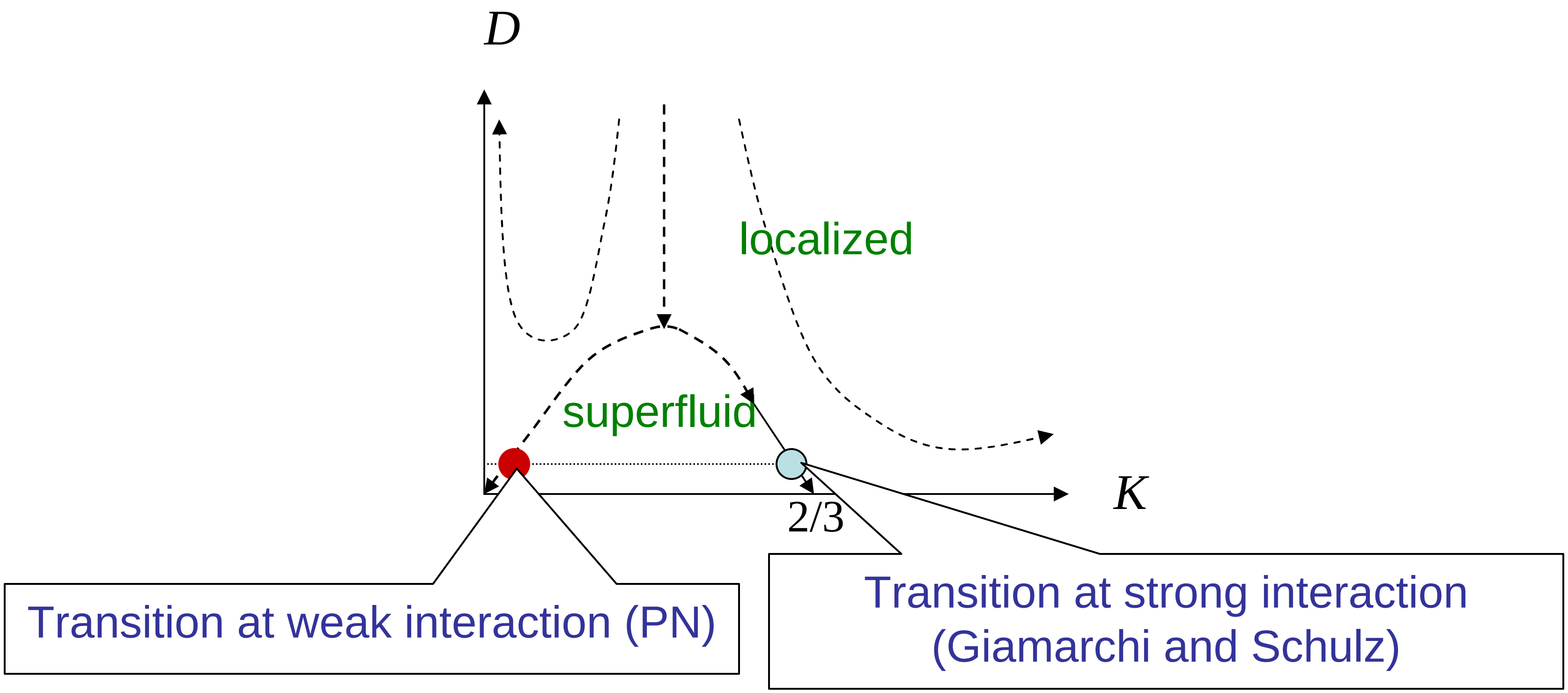
At $n=n_c$ quantum phase transition from disordered singlet state to disordered superfluid. Scaling near transition was studied by Matthew Fisher *et al*. Marginal dimensionality is 1 (Giamarchi and Schulz). Is it the same transition?

Model description: random quantum *XY*-model:

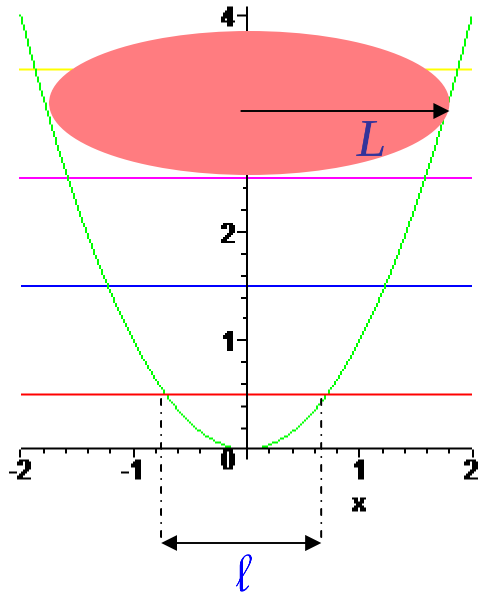
$$H_{XY} = \sum_j K_j \left(N_j - \bar{N}_j\right)^2 - \sum_{ij} t_{ij} \cos\left(\varphi_i - \varphi_j\right)$$

$n \gg n_c$ -- almost homogeneous superfluid, small corrections to Bogolyubov formulae from disorder (Huang and Meng)

Giamarchi-Schulz phase diagram



Weakly repulsive gas in a harmonic trap



Single particle states in a trap

$$V(\mathbf{x}) = \frac{m\omega^2 \mathbf{x}^2}{2}$$

Energy levels: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$\ell = \sqrt{\frac{\hbar}{m\omega}}$ Magnetic length

Weakly repulsive N particles in a harmonic trap, no disorder, form a spherical cloud of the size R

$$E(R, N) = N\left(\frac{\hbar^2}{2mR^2} + \frac{m\omega^2 R^2}{2} + gn\right)$$

$n = \frac{3N}{4\pi R^3}$; $g = \frac{4\pi\hbar^2 a}{m}$

Kinetic energy — *Trap energy* — *Interaction energy*

$$E(L,N) = N\frac{\hbar^2}{2m}\left(\frac{1}{L^2} + \frac{L^2}{\ell^4} + \frac{3Na}{L^3}\right)$$

Must be minimized over *R*

Weak interaction $3Na \square \ \ell \Rightarrow L = \ell$

Strong interaction $3Na \square \ \ell \Rightarrow L = \left(\frac{9}{2}Na\ell^4\right)^{1/5}$

Thomas-Fermi approximation

Disorder potential $\langle U(\mathbf{x})U(\mathbf{x}')\rangle = \kappa^2 \delta(\mathbf{x}-\mathbf{x}')$

$\mathfrak{L} \square \ \ell$ -- weak disorder; small corrections to the previous results

Strong disorder, weak interaction

$a \square \ Na \square \ \mathfrak{L} \square \ \ell$ -- *non-ergodic situation*; in equilibrium all particles occupy a single random potential well with minimal energy

Size of the cloud is determined by the energy balance:

$$\mathcal{E}_{\mathfrak{L}} = \frac{\hbar^2}{2m\mathfrak{L}^2} \square \frac{m\omega^2 L^2}{2}$$

$$L = \frac{\ell^2}{\mathfrak{L}}$$

Energy of disorder
(logarithmic accuracy)

Energy of trap

Typical size of the well: $R = \dfrac{\mathfrak{L}}{6\ln(\ell/\mathfrak{L})}$

Strong disorder, moderately strong interaction: $a \ll \mathfrak{L} \ll Na \ll \ell$

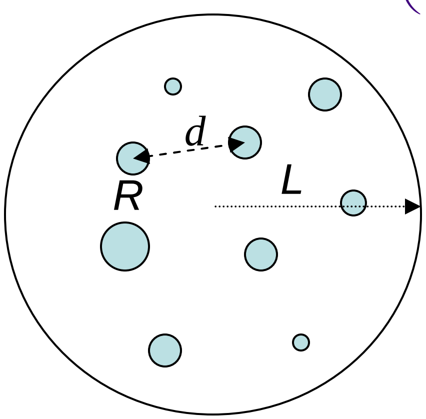
Ergodic situation

$N \gg \left(\ell^6 / \mathfrak{L}^5 a\right)$ or $\Gamma = \ell^6 / \left(3\mathfrak{L}^5 aN\right) \ll 1$

Multiple random potential wells (fragmentation). Average particle density n is a slow function of coordinate r

$$\mu = gn_p(r) - \frac{\hbar^2}{2mR^2(r)} + \frac{m\omega^2 r^2}{2} \approx -\mathcal{E}_{\mathfrak{L}} \ln^2 \frac{n_c}{n(r)} + \frac{m\omega^2 r^2}{2}$$

$$n(r) = n_c \exp\left(-\sqrt{\frac{\mathfrak{L}^2 r^2}{\ell^4} - \frac{\mu}{\mathcal{E}_{\mathfrak{L}}}}\right) \approx n_c \exp\left(-\sqrt{-\mu/\mathcal{E}_{\mathfrak{L}}}\right) \exp\left(-\frac{\mathfrak{L}^2 r^2}{2\ell^4 \sqrt{-\mu/\mathcal{E}_{\mathfrak{L}}}}\right)$$



$$R = \mathfrak{L}/\ln\Gamma \qquad d = \mathfrak{L}\Gamma^{1/3}/\ln\Gamma \qquad L = \left(\ell^2/\mathfrak{L}\right)\sqrt{\ln\Gamma}$$

No coherence, random singlet or Bose glass

Counter-intuitive dependence of L on N

$$N \gg \left(\ell^6/\mathfrak{L}^5 a\right) \qquad \text{or} \qquad \Gamma = \ell^6/\left(3\mathfrak{L}^5 aN\right) \gg 1$$

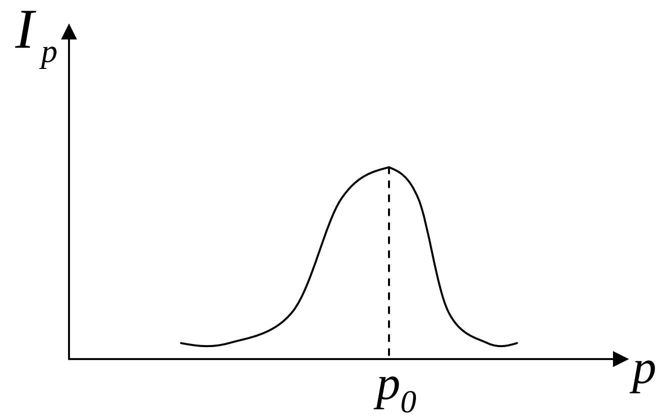
Small corrections of the order of [illegible] to the state without disorder

Superfluid

Phase transition line: $\Gamma = 1$

Distribution of momentum in fragmented state

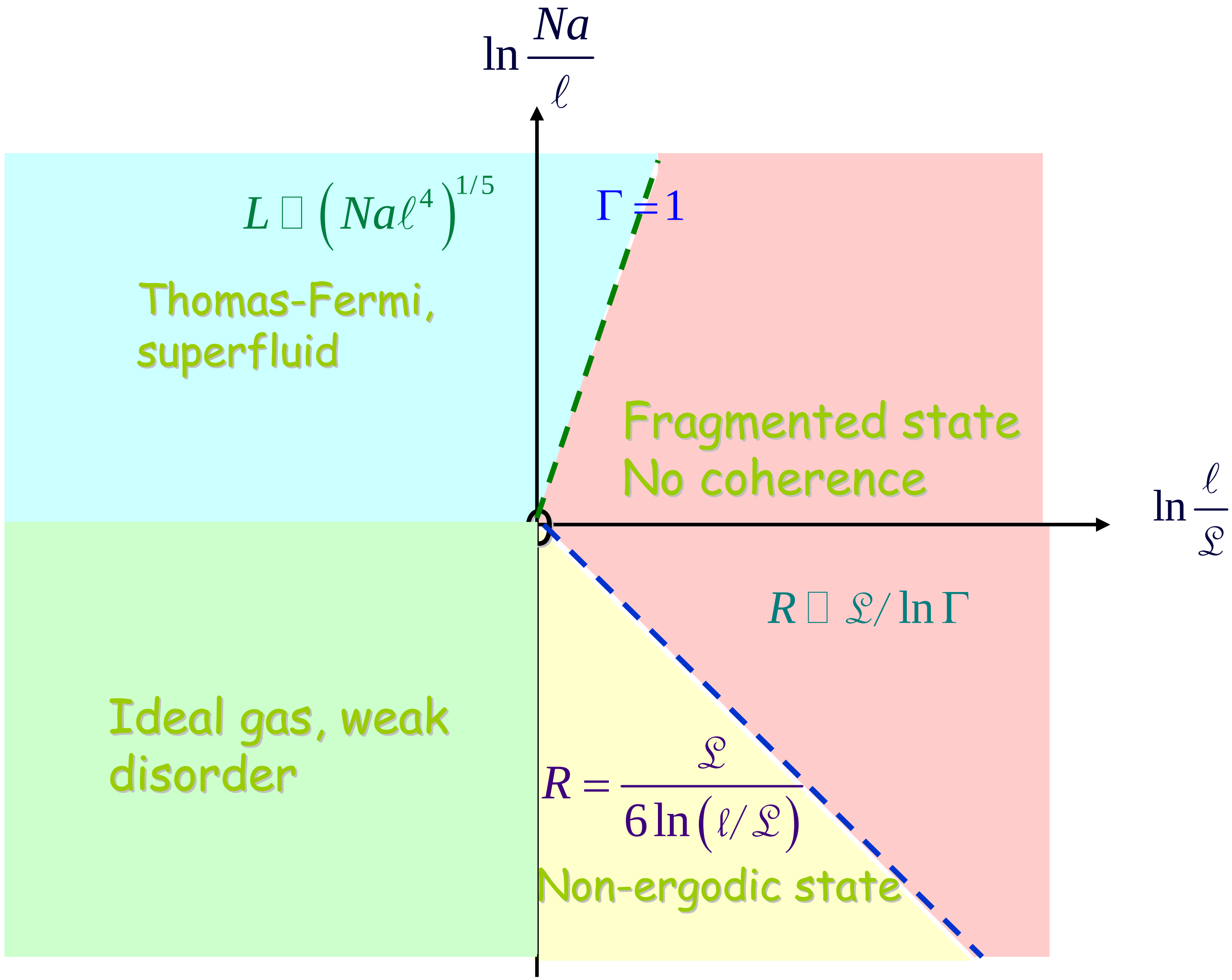
$$p_0 = \left(\hbar/\mathfrak{L}\right)\ln\Gamma$$



Number of fragments

$$N_F \sim \frac{12Na}{\mathfrak{L}}\left(\ln\Gamma\right)^{9/2}$$

Phase diagram



Conclusions

- We describe properties of weakly interacting Bose gas in a random Gaussian potential with zero correlation length and approach (but not reach) the superfluid transition line.
- The character and quantitative characteristics of weakly interacting Bose-gas in such potential can be described in terms of two characteristic lengths: the Larkin length characterizing disorder and the scattering amplitude
- The ground state of ideal Bose gas in random potential in thermodynamic limit is non-ergodic
- Even a weak interaction changes drastically the property of the ground state
- At average particle density much less than a critical one, the Bose particles occupy deep potential wells and exponentially weakly tunnel to other wells. The number of particles in each well is defined, but phases are uncertain. It is a disordered singlet state.
- At average particle density equal to the critical value determined by interaction and disorder, the transition to the inhomogeneous superfluid proceeds.
- In a trap, at sufficiently strong disorder and weak interaction (small value of the product of number of particles and the scattering amplitude), the ground state is non-ergodic.
- Starting from a critical value of the above mentioned product the state becomes ergodic and fragmented. There is no coherence between fragments.
- At a fixed disorder the growing number of particles leads to transition from disordered singlet to superfluid state.
- In terms of 1d theory by Giamarchi and Schulz we describe the approach to the phase transition at weak interaction, which they conjectured but could not treat quantitatively.

What did not enter to this presentation

- Two- and one- dimensional traps
- Bose-gases with attraction in a disordered trap
- Fermi-gases
- Dipolar oscillations of the clouds in traps
- Single-connected states at very large number of particles due to the trap potential
- Correlated noise with correlation length larger than Larkin length

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