23 June 2008
Landau100

δN formalism and nonlinear curvature perturbations from inflation

Misao Sasaki

Yukawa Institute (YITP)
Kyoto University

1. Introduction

- scales in cosmology
- slow-roll inflation
- number of e-folds of inflation

2. Cosmological perturbation theory

- metric perturbation & time slicing
- δN formalism in linear theory

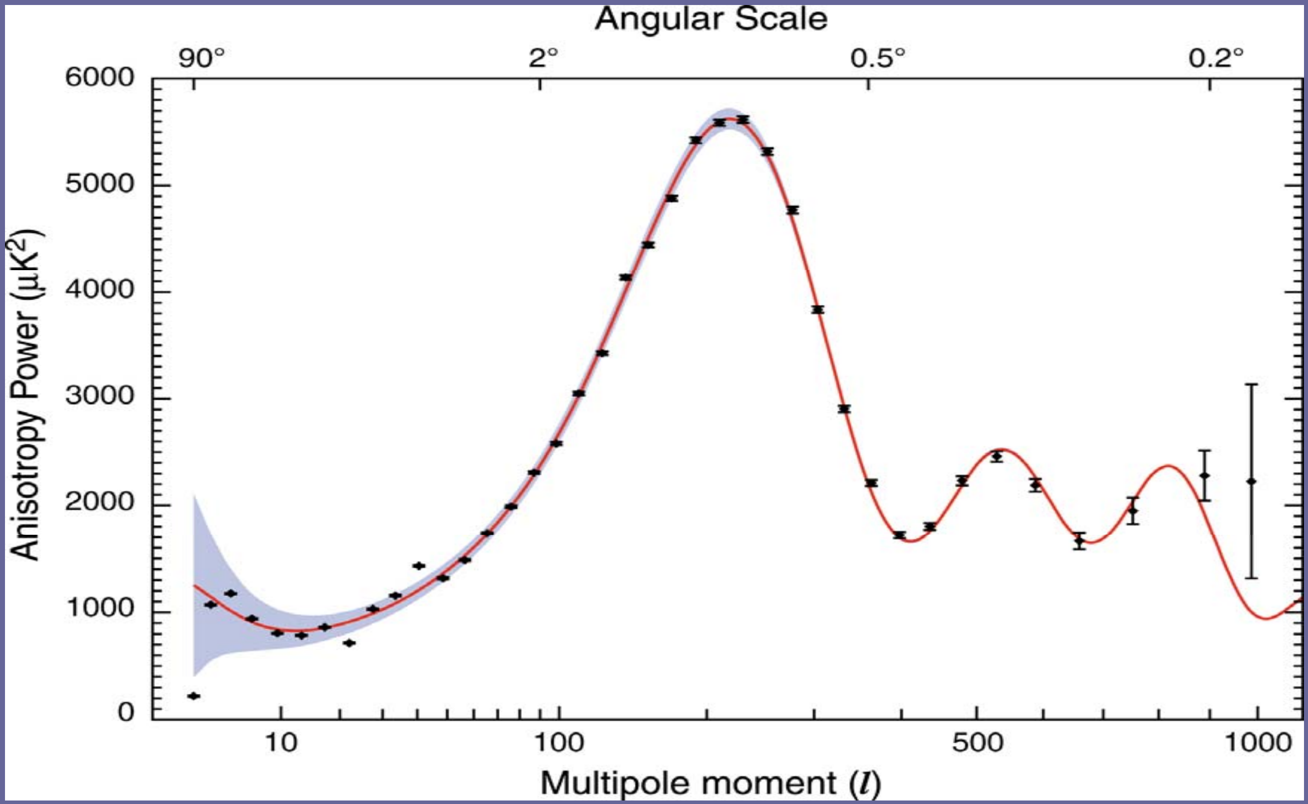
3. Nonlinear extension

- separate universe approach
- diagrammatic method for nonlinear δN

4. Summary

1. Introduction

- Standard (single-field, slowroll) inflation predicts scale-invariant Gaussian curvature perturbations.



- CMB (WMAP) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

However, nature may not be so simple...

- Tensor perturbations (gravitational waves) have not been detected yet.

  tensor-scalar ratio: $r<0.2$ (95%CL)

  WMAP+BAO+SN ('08)

- Future CMB experiments may detect non-Gaussianity

  gravitational potential: $\Phi = \Phi_{gauss} + f_{NL}\Phi^2_{gauss} + \cdots$

  $-9 < f_{NL} < 111$ (95%CL) WMAP 5yr ('08)

- Inflationary universe models need to be tested.

  multi-field, non-slowroll, extra-dim's, string theory...

δN formalism for curvature perturbations

Scales in cosmology

$a(t)$: cosmic scale factor (~ relative size of the universe)

$$H = \frac{\dot{a}}{a}: \text{ expansion rate}$$

H^{-1}: expansion time scale (~ age of the universe)

$cH^{-1} = H^{-1}$ (in units of $c = 1$): Hubble horizon radius
(size of causally connected region)

$L < H^{-1}$: sub-horizon scale
(scale of everyday physics)

$L > H^{-1}$: super-horizon scale
(no propagation of information;
spatial variations are frozen)

Slow-roll inflation

Linde '82

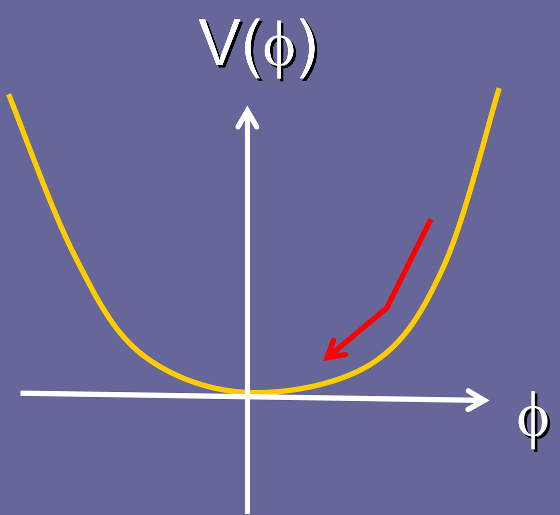
- single-field inflation, no other degree of freedom

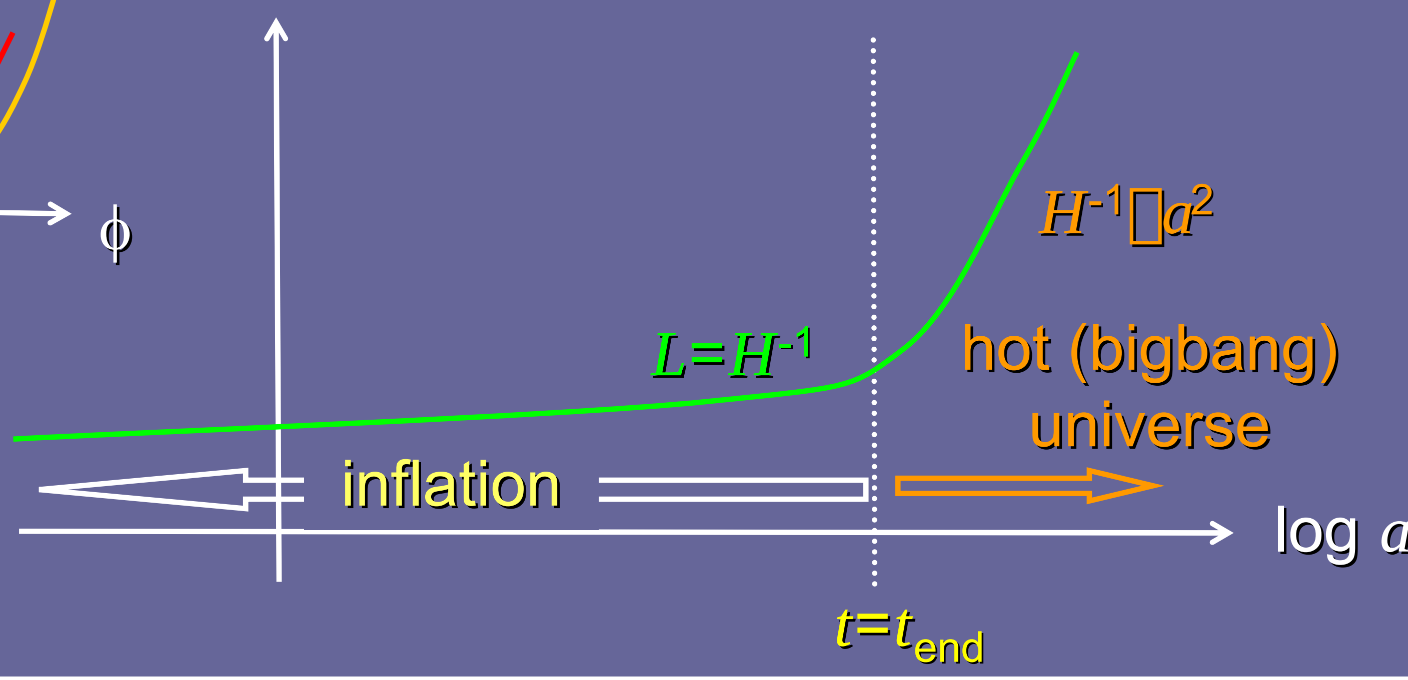
$$H^2 = \frac{8\pi G}{3}\left[\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi)\right], \quad \cancel{\ddot{\phi}} + 3H\dot{\phi} + V'(\phi) = 0 \quad \Rightarrow \quad t = t(\phi)$$

(friction-dominated) over-damped system

$H \approx$ const.

~ exponential expansion

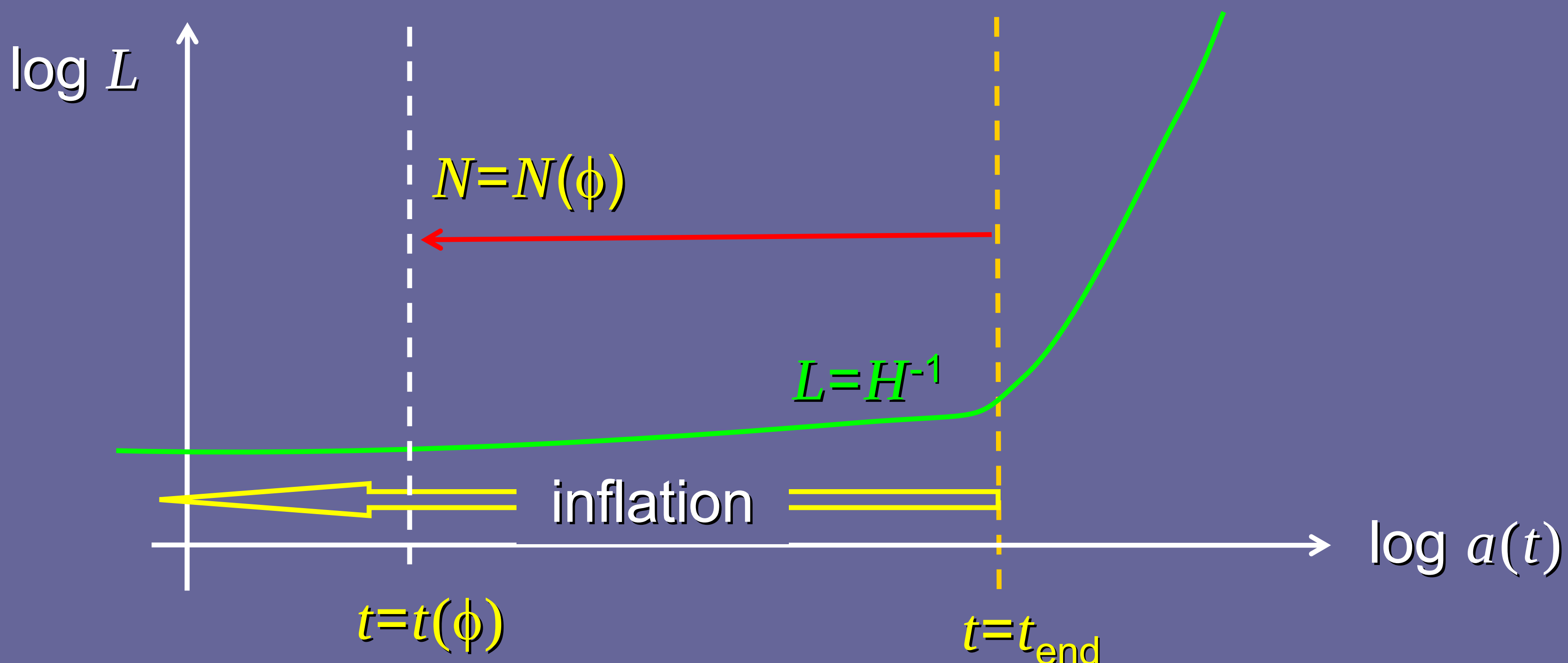




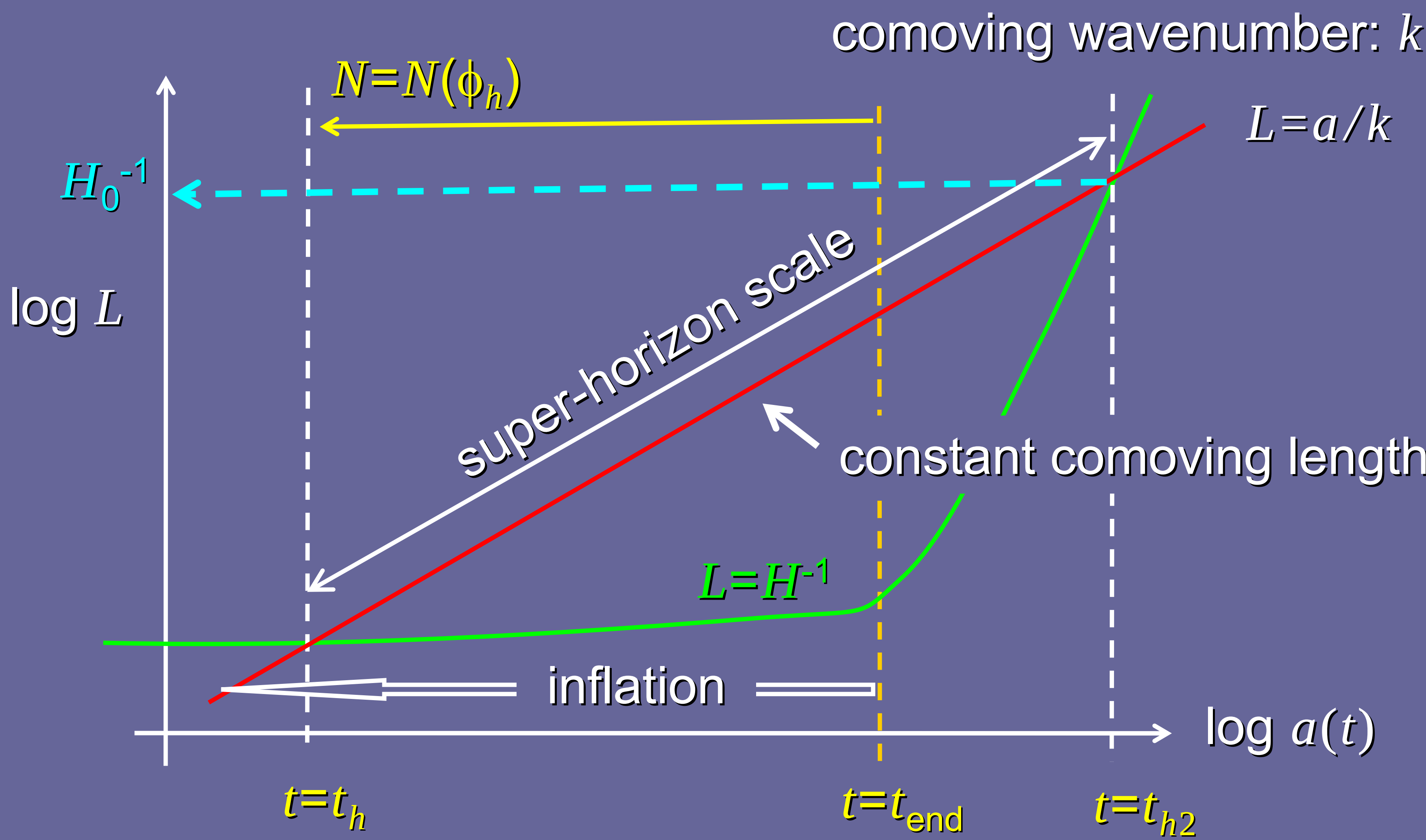
Number of e-folds of inflation

• number of e-folds counted *backward in time* from the end of inflation

$$\frac{a(t_{\text{end}})}{a(t)} = \exp[N(t \to t_{\text{end}})] \Rightarrow N = N(\phi) = \int_{t(\phi)}^{t_{\text{end}}} H dt$$



Comoving scales



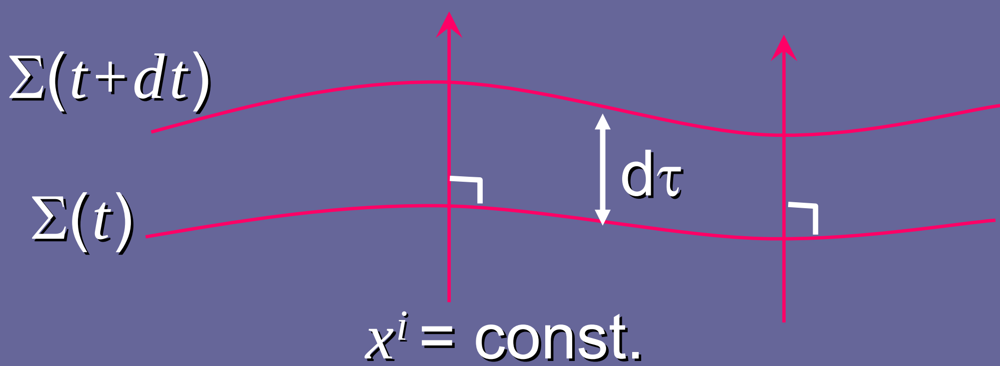
- present Hubble horizon radius □$N=N(\phi_h)$ ~ 60
- perturbation freezes out on superhorizon scales

2. Cosmological perturbation theory

Lifshitz '63, ...

- metric on a spatially flat background (g_{0j}=0 for simplicity)

$$ds^2 = -(1+2A)dt^2 + a^2(t)\left[(1+2\mathrm{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \partial_i \partial_j E$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

- propertime along x^i = const.: $d\tau = (1+A)dt$
- curvature perturbation on $\Sigma(t)$: P $\Leftrightarrow$ $\overset{(3)}{R} = -\dfrac{4}{a^2}\overset{(3)}{\Delta}\mathrm{R}$
- expansion (Hubble parameter): $\tilde{H} = H(1-A) + \partial_t\left[\mathrm{R} + \dfrac{1}{3}\overset{(3)}{\Delta}E\right]$

a: cosmic scale factor $H \equiv \dot{a}/a$

Typical choices of time-slicing

- comoving slicing $\quad T^{\mu}{}_{i} = 0$ (~matter rest frame)

matter-based slices

- uniform density slicing $\quad -T^{0}{}_{0} \equiv \rho = \rho(t)$

- uniform Hubble slicing

$$\tilde{H} = H(t) \quad \Leftrightarrow \quad -H\,A + \partial_t \left[\mathcal{R} + \frac{1}{3} {}^{(3)}\Delta\, E \right] = 0$$

geometrical slices

- flat slicing $\quad {}^{(3)}R = -\frac{4}{a^2} {}^{(3)}\Delta\, \mathcal{R} = 0 \Leftrightarrow \mathcal{R} = 0$

comoving $\approx$ uniform ρ $\approx$ uniform H

δN formalism in linear theory

MS & Stewart '96

e-folding number perturbation between $\Sigma(t)$ and $\Sigma(t_{\rm fin})$:

$$\delta N\left(t;t_{\rm fin}\right) \equiv \int_t^{t_{\rm fin}} \tilde{H} d\tau - \left(\int_t^{t_{\rm fin}} H\, d\tau\right)_{\rm background}$$

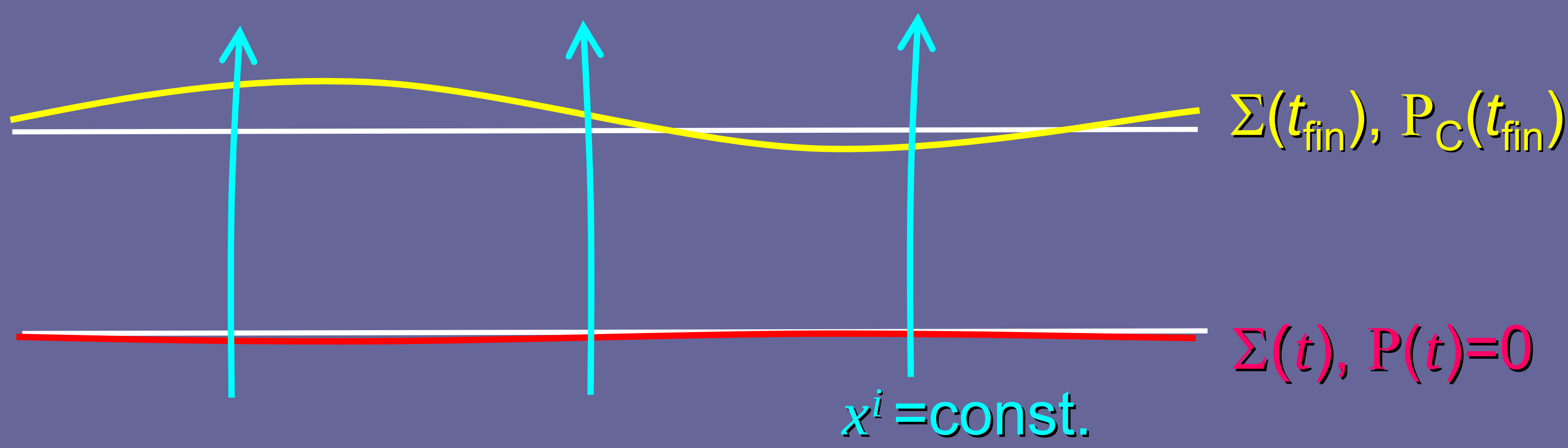
$$= \int_t^{t_{\rm fin}} \partial_t \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} E \right] dt = \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} E \right]_t^{t_{\rm fin}}$$

spatial gradients are negligible on superhorizon scales

depends only on initial and final time

$\delta N(t;t_{\rm fin}) = \mathcal{R}(t_{\rm fin}) - \mathcal{R}(t)$ on superhorizon scales

Choose $\Sigma(t)$ = flat (P=0) and $\Sigma(t_{\rm fin})$ = comoving / uniform density:



⇨ $\delta N(t;t_{\rm fin}) = \mathcal{R}_C(t_{\rm fin})$ on superhorizon scales

↑ curvature perturbation on comoving slice
(suffix 'C' for comoving)

$\mathcal{R}_C(t_{\rm fin})$ ··· directly related to gravitational potential perturbation in the present universe

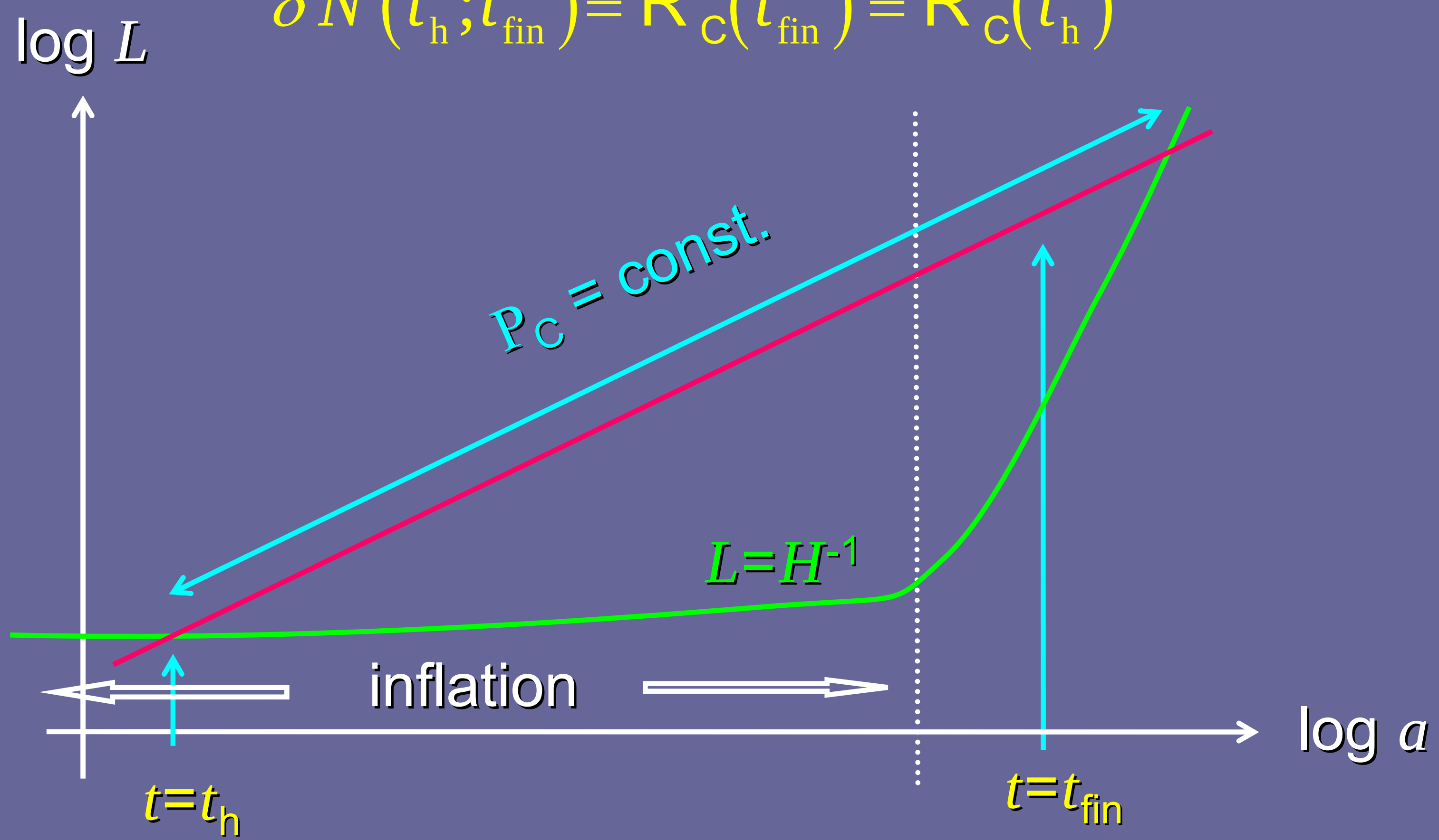
By definition, $\delta N(t;\, t_{\rm fin})$ is t-independent

δN in single-field slowroll inflation

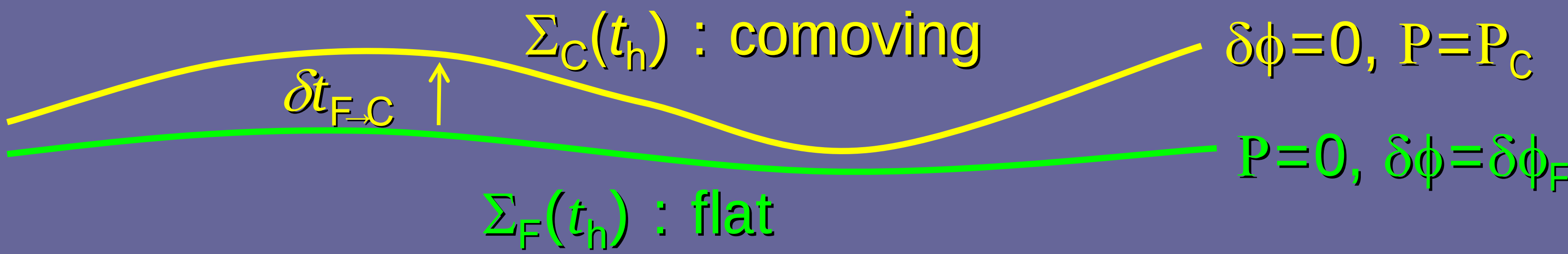
Starobinsky '82

P_C becomes constant soon after horizon-crossing ($t=t_h$):

$$\delta N\left(t_h;t_{fin}\right)=\mathrm{R}_C\left(t_{fin}\right)=\mathrm{R}_C\left(t_h\right)$$



Also $\delta N = H(t_{\rm h})\,\delta t_{F\to C}$, where $\delta t_{F\to C}$ is the time difference between the comoving and flat slices at $t=t_{\rm h}$.



$$\phi_{\rm F}\left(t_{\rm h}+\delta t_{\rm F\to C},x^i\right)=\phi_{\rm C}\left(t_{\rm h}\right) \Rightarrow \delta\phi_{\rm F}+\dot{\phi}\left(t_{\rm h}\right)\delta t_{\rm F\to C}=0$$

$$\Rightarrow \mathrm{R}_{\rm C}\left(t_{\rm fin}\right)=\delta N\left(t_{\rm h};t_{\rm fin}\right)=-\frac{H}{d\phi/dt}\delta\phi_{\rm F}\left(t_{\rm h}\right) \Leftarrow dN=-H\,dt$$

$$=\frac{dN}{d\phi}\delta\phi_{\rm F}\left(t_{\rm h}\right) \quad \cdots \ \delta N \text{ formula}$$

Starobinsky '82

- δN for multi-component inflation MS & Stewart '96

$$\mathcal{R}_C(t_{\rm fin}) = \delta N = N(\phi + \delta\phi_F) - N(\phi) = \sum_a \frac{\partial N}{\partial \phi^a}\delta\phi_F^a(t_h)$$

Only the knowledge of the background evolution is necessary to calculate $P_C(t_{\rm fin})$.

each horizon size region evolves independently because of causality.

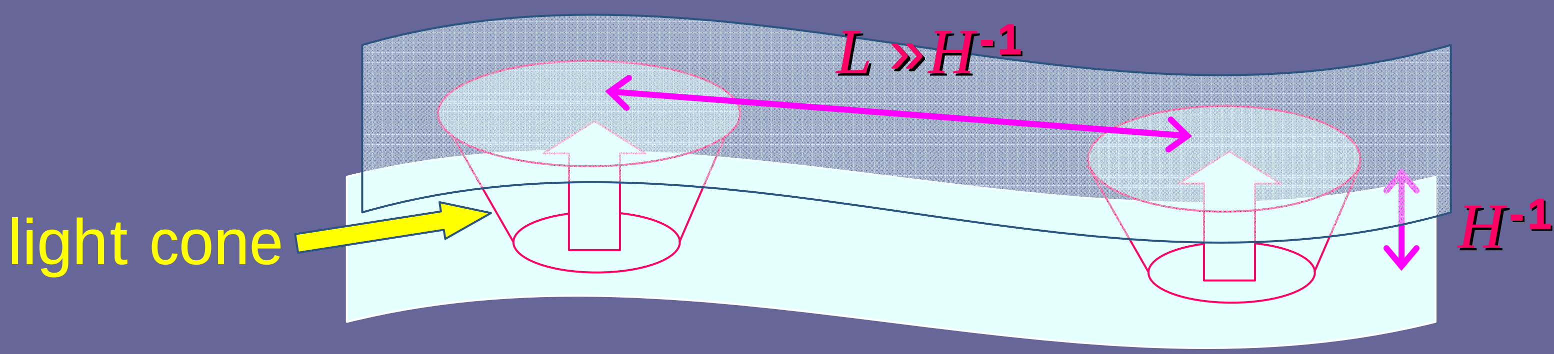
3. Nonlinear extension

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i}Q\right| \ll \left|\frac{\partial}{\partial t}Q\right| \sim HQ; \;\; H \sim \sqrt{G\rho}$$

Belinski Khalatnikov & Lifshitz '70,
Tomita '72, Salopek & Bond '90, ...

This is a consequence of causality:



- At lowest order, no signal propagates in spatial directions.

Field equations reduce to ODE's

Separate Universe Approach

- Each horizon size region evolves independently as a homogeneous universe

⇩

δN formula is applicable also to nonlinear perturbations on superhorizon scales

MS & Tanaka '98, Lyth, Malik &MS '05

- Nonlinear δN formula for multi-component inflation:

$$\delta N = N(\phi+\delta\phi) - N(\phi) = \sum_n \frac{1}{n!}\frac{d^n N}{d\phi^{a_1}\cdots d\phi^{a_n}}\delta\phi^{a_1}\cdots\delta\phi^{a_n}$$

Diagrammatic method for nonlinear δN

Byrnes, Koyama, MS & Wands '07

$$\zeta \equiv \delta N = \sum_n \frac{N_{A_1 A_2 \cdots A_n}}{n!}\,\delta\phi^{A_1}\delta\phi^{A_2}\cdots\delta\phi^{A_n}\ ; \qquad N_{A_1 A_2 \cdots A_n} \equiv \frac{D^n N}{\partial\phi^{A_1}\partial\phi^{A_2}\cdots\partial\phi^{A_n}}$$

'basic' 2-pt function: $\left\langle \delta\phi^A(x)\delta\phi^B(y)\right\rangle = h^{AB}(\phi)G_0(x-y)$ ← field space metric

Here $\delta\phi$ is assumed to be Gaussian.

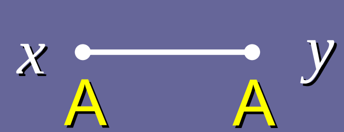
Extension to non-Gaussian $\delta\phi$ is straightforward.

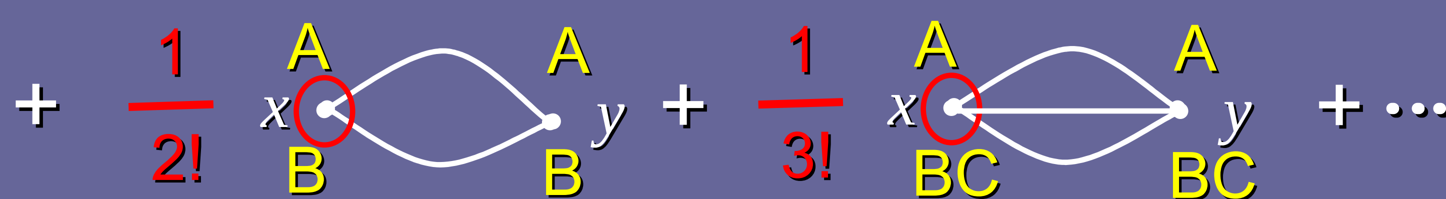
- connected n-pt function of ζ:

2-pt function

$$\left\langle \zeta(x)\zeta(y)\right\rangle_c = N_A N^A G_0(x-y) + \frac{1}{2!}N_{AB}N^{AB}G_0(x-y)^2 + \frac{1}{3!}N_{ABC}N^{ABC}G_0(x-y)^3 + \cdots$$

⟺





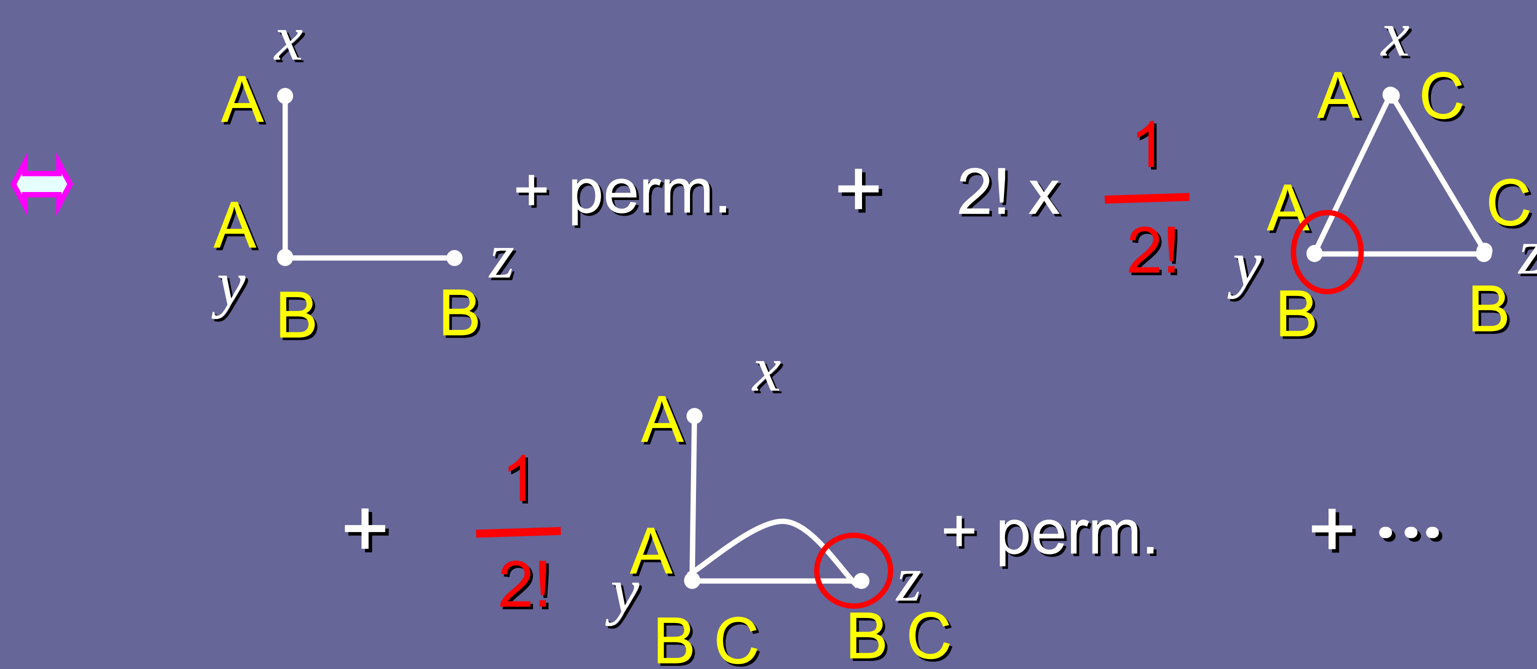
3-pt function

$$\left\langle \zeta(x)\zeta(y)\zeta(z)\right\rangle_c = N^{A}N_{AB}N^{B}G_0(x-y)G_0(y-z)+\text{perm.}$$

$$+N^{AB}N_{BC}N^{CA}G_0(x-y)G_0(y-z)G_0(z-x)$$

$$+\frac{1}{2!}N^{A}N_{ABC}N^{BC}G_0(x-y)^2G_0(y-z)+\text{perm.}$$

$$+\text{L}$$



4. Summary

- Superhorizon scale perturbations are always local, in the sense that no propagation effect is important.

  separate universe

  →curvature perturbation can be evaluated by only the knowledge of the background evolution: δN formula

  N.B., nonlocal nonlinearity (non-Gaussianity) may appear due to quantum interactions on subhorizon scales.

  eg, DBI (brane) inflation

- Nonlinear δN formula is a very useful tool in evaluating non-Gaussianity from inflation.

  diagrammatic method

  →systematic evaluation of non-Gaussianity