

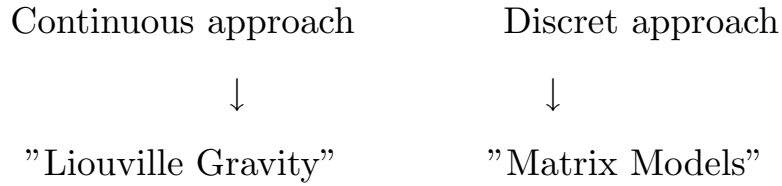
**On the correlation numbers
in Minimal Gravity and Matrix Models**

A.Belavin, A.Zamolodchikov

Landau 100
Chernogolovka, June 2008

Summary

Two approaches to 2D quantum geometry



Impressive body of evidence that the two describe the same reality:

- Operators

$$O_k^{LG}$$

$$O_k^{MM}$$

have identical scale dimensions

- Some correlation numbers coincide:

$$\langle O_1^{LG} \dots O_n^{LG} \rangle = \langle O_1^{MM} \dots O_n^{MM} \rangle$$

But with ”naive” identification many correlation numbers are not in agreement.

Resolution [*Moore, Seiberg, Staudacher, 1991*]: Resonance relations:

$$[O_k] = [\tau_{k_1}][O_{k_2}]$$

↓

$$\text{Umbiguity } O_k^{MM} = O_k^{LG} + B_k^{k_1 k_2} \tau_{k_1} O_{k_2}^{LG}$$

- In many cases the disagreement can be fixed by adjusting the parameters (e.g. $B_k^{k_1 k_2}$ above)

- This work: Trying to find exact map for special class of models:

$$\text{"Minimal Gravity" } \mathcal{MG}_{2/2p+1} \quad \leftrightarrow \quad \begin{array}{l} \text{"p - criticality" in} \\ \text{One - Matrix Model} \end{array}$$

- The problem is rather "rigid" (more constraints than the parameters).
- Nonetheless, the map exists up to the level of four point corr. numbers.
- The resulting 1-, 2-, 3-, and 4-point correlation numbers are in perfect agreement.

1. Minimal Gravity

1.1. Quantum Geometry

$$\sum_{\text{topologies}} \int D[g] D[\phi] e^{-S[g,\phi]}$$

$g(x)$ - Riemannian metric on 2D manifold \mathbb{M} (assume sphere)

ϕ - "matter" fields

Invariant correlation functions ("correlation numbers") :

$$\langle \tilde{O}_{k_1} \dots \tilde{O}_{k_N} \rangle = Z^{-1} \int \tilde{O}_{k_1} \dots \tilde{O}_{k_N} e^{-S[g,\phi]} D[g, \phi]$$

with

$$\tilde{O}_k = \int_{\mathbb{M}} O_k(x) d\mu_g(x)$$

$O_k(x)$ - local fields (built from ϕ and g).

Generating function: $\{\tau\} = \{\tau_1, \dots, \tau_n\}$

$$W(\{\tau\}) = Z(\{\tau\})/Z(\{0\}), \quad Z(\{\tau\}) = \int D[g, \phi] e^{-S_\tau[g,\phi]},$$

$$S_\tau[g, \phi] = S_0[g, \phi] + \sum_k \tau_k \tilde{O}_k$$

so that

$$\langle \tilde{O}_{k_1} \dots \tilde{O}_{k_N} \rangle = \left. \frac{\partial^N W(\{\tau\})}{\partial \tau_{k_1} \dots \partial \tau_{k_N}} \right|_{\tau=0}$$

The parameters $\{\tau\}$ may be regarded as the coordinates in the "theory space" Σ .

1.2. Conformal Matter, and Liouville Gravity

$$g^{\mu\nu} T_{\mu\nu}^{\text{matter}} = -\frac{c}{12} R$$

Conformal Gauge $g_{\mu\nu} = e^{2b\varphi} \hat{g}_{\mu\nu}$: \Rightarrow Decoupling

$$S[g, \phi] \rightarrow S_L[\varphi] + S_{\text{Ghost}}[B, C] + S_{\text{Matter}}[\phi]$$

with

$$S_L[\phi] = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[\hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + Q \hat{R} \varphi + 4\pi\mu e^{2b\varphi} \right] d^2x,$$

$$S_{\text{Ghost}}[B, C] = \frac{1}{2\pi} \int \sqrt{\hat{g}} B_{\mu\nu} \nabla^\mu C^\nu d^2x,$$

$$(B_{\mu\nu} = B_{\nu\mu}, \quad \hat{g}^{\mu\nu} B_{\mu\nu} = 0),$$

$$26 - c = 1 + 6Q^2 \quad Q = b + 1/b.$$

($S_{\text{Matter}}[\phi]$ is conformally invariant, with the central charge c).

Correlation numbers $\langle \tilde{O}_{k_1} \dots \tilde{O}_{k_N} \rangle$ with

$$\tilde{O}_k = \int V_k(x) \Phi_k(x) d^2x$$

$\Phi_k(x)$ - (spinless) primary fields of the matter CFT, with the conformal dimensions (Δ_k, Δ_k)

$V_k(x)$ - "gravitational dressings",

$$V_k(x) = e^{2a_k \varphi(x)}, \quad a_k(Q - a_k) + \Delta_k = 1$$

Gravitational dimensions of \tilde{O}_k control the scale dependence of the corr. functions:

$$\tilde{O}_k \sim \mu^{\delta_k}, \quad \delta_k = -\frac{a_k}{b}$$

1.3. Correlation numbers

$$\begin{aligned} \langle \tilde{O}_{k_1} \dots \tilde{O}_{k_n} \rangle &= |(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^2 \times \\ &\int d^2 x_4 \dots d^2 x_n \underbrace{\langle O_{k_1}(x_1) O_{k_2}(x_2) O_{k_3}(x_3) O_{k_4}(x_4) \dots O_{k_n}(x_n) \rangle}_{\downarrow} \\ &\langle V_{k_1}(x_1) \dots V_{k_n}(x_n) \rangle_{\text{Liouville}} \langle \Phi_{k_1}(x_1) \dots \Phi_{k_n}(x_n) \rangle_{\text{Matter}} \end{aligned}$$

$$(V_k(x) = e^{2a_k \varphi(x)}, \quad a_k(Q - a_k) = 1 - \Delta_k)$$

- The Liouville correlation functions are expressed in terms of the "Conformal Blocks", e.g.

$$\begin{aligned} \langle V_{k_1}(x_1) \dots V_{k_4}(x_4) \rangle_{\text{Liouville}} &= \\ \int \frac{dP}{4\pi} C_L(a_{k_1}, a_{k_2}, Q/2 + iP) C_L(Q/2 - iP, a_{k_3}, a_{k_4}) &|\mathcal{F}_{\Delta(P)}(1 - \Delta_i | x_i)|^2 \end{aligned}$$

with $\Delta(P) = Q^2/4 + P^2$, and the "Liouville Structure Constants"

$$C_L(a_1, a_2, a_3) = (\pi\mu\gamma(b^2))^{(Q-a)/b} \frac{\Upsilon_b(b)}{\Upsilon_b(a-Q)} \prod_{i=1}^3 \frac{\Upsilon_b(2a_i)}{\Upsilon_b(a-a_i)}$$

where $a = a_1 + a_2 + a_3$,

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\frac{(Q-2x)^2}{4} e^{-2t} - \frac{\sinh^2((Q/2-x)t)}{\sinh(bt) \sinh(t/b)} \right]$$

- Integration over the moduli x_4, \dots, x_n is to be performed.

1.4. Matter CFT: "Minimal Models"

$$\mathcal{M}_{p/q} \quad c = 1 - 6 \frac{(p-q)^2}{pq}$$

Finite number of primary fields

$$\Phi_{(n,m)} \quad (n = 1, \dots, p-1, \quad m = 1, \dots, q-1, \quad n \leq m),$$

with (in principle) computable correlation functions, e.g.

$$\begin{aligned} \langle \Phi_{(n_1, m_1)}(x_1) \dots \Phi_{(n_4, m_4)}(x_4) \rangle_{MM} = \\ \sum_{(n,m)} \mathbb{C}_{(n_1, m_1)(n_2, m_2)}^{(n,m)} \mathbb{C}_{(n_3, m_3)(n_4, m_4)}^{(n,m)} |\mathcal{F}_{(n,m)}(\Delta_i | x)|^2 \end{aligned}$$

Fusion rules:

$$\Phi_{(n_1, m_1)} \Phi_{(n_2, m_2)} = \sum_{n=|n_1-n_2|+1}^N \sum_{m=|m_1-m_2|+1}^M [\Phi_{(n,m)},$$

with

$$\begin{aligned} N &= \min(n_1 + n_2 - 1, 2p - n_1 - n_2 - 1), \\ M &= \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1) \end{aligned}$$

1.5. "Yang-Lee series" of the Minimal Models $\mathcal{M}_{2/2p+1}$

- $\mathcal{M}_{2/2p+1}$ has p primary fields

$$\Phi_k \equiv \Phi_{(1,k+1)}, \quad k = 0, 1, \dots, p-1 \quad (p, p+1, \dots, 2p-1)$$

Fusion rules

$$[\Phi_{k_1}][\Phi_{k_2}] = \sum_{k=|k_1-k_2|:2}^{k_1+k_2} [\Phi_k], \quad [\Phi_k] = [\Phi_{2p-k-1}]$$

$$\text{"Parity"} : \quad \Phi_k \quad \begin{cases} + \text{ for even } k \\ - \text{ for odd } k \end{cases}$$

$$\Phi_k = \Phi_{2p-k-1} \quad \rightarrow \quad \text{"Parity violation"}$$

- Correlation functions:

$$\langle \Phi_k \rangle = \delta_{k,0}, \quad \langle \Phi_k \Phi_{k'} \rangle \sim \delta_{k,k'}$$

$$\langle \Phi_{k_1} \Phi_{k_2} \Phi_{k_3} \rangle = 0 \quad \text{if} \quad \begin{cases} k_1 + k_2 < k_3, \text{ etc,} & \text{for } k_1 + k_2 + k_3 \text{ even} \\ k_1 + k_2 + k_3 < 2p-1 & \text{for } k_1 + k_2 + k_3 \text{ odd} \end{cases}$$

$$\langle \Phi_{k_1} \dots \Phi_{k_n} \rangle = 0 \quad \text{if} \quad \begin{cases} k_1 + \dots + k_{n-1} < k_n, & \text{for } k_1 + \dots + k_n \text{ even} \\ k_1 + \dots + k_n < 2p-1 & \text{for } k_1 + \dots + k_n \text{ odd} \end{cases}$$

- Interpretations:

$\mathcal{M}_{2/3}$ - "empty" theory (has only identity operator)

$\mathcal{M}_{2/5}$ - Yang-Lee edge criticality [*Cardy, 1985*]

$\mathcal{M}_{2/2p+1}$ - Yang-Lee multi-criticality?

1.6. Minimal gravity $\mathcal{MG}_{p/q}$:

$\mathcal{M}_{p/q}$ coupled to the Liouville Gravity

- Early computations of the correlation numbers: [*Goulian & Li, 1991; Di Francesco & Kutasov, 1991; ...*]

- Systematic approach [*Alexei Zamolodchikov, 2004; Belavin & Al.Zamolodchikov, 2006*]:

”Higher Liouville Equations of Motion”

$$\int_{\text{moduli}} [\dots] = \int_{\text{moduli}} [\text{total derivative}] \rightarrow \text{Boundary terms}$$

- Results for $\mathcal{MG}_{2/2p+1}$:

$$\tilde{O}_k = \int V_k(x) \Phi_k(x) d^2x, \quad V_k(x) = e^{(k+2)b\varphi(x)}$$

with

$$b = \sqrt{2/(2p+1)}$$

★ One-point correlation numbers

$$\langle \mathcal{O}_k \rangle = 0,$$

★★ Two-point numbers

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle = \frac{\delta_{kk'}}{Z_p} \frac{1}{2p-2k-1} \text{Leg}_L^2(k),$$

with

$$Z_p = [(2p-1)(2p+1)(2p+3)]^{-1}$$

and

$$\text{Leg}_L(k) = \frac{[\pi\mu\gamma(\frac{2}{2p+1})]^{-\frac{k+2}{2}}}{2p-1} \left[\frac{\pi^2 \gamma(\frac{2}{2p+1})\gamma(\frac{2p+1}{2})}{\gamma(\frac{2p-2k-1}{2p+1})\gamma(\frac{2p-2k-1}{2})} \right]^{1/2}$$

*** Three-point correlation numbers:

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \rangle = \frac{N_{k_1 k_2 k_3}}{Z_p} \prod_{i=1}^3 \text{Leg}_L(k_i)$$

where $N_{k_1 k_2 k_3}$ enforces the "fusion rules"

$$N_{k_1 k_2 k_3} = \begin{cases} 1 & \text{if the fusion rules of } \mathcal{M}_{2/2p+1} \text{ are satisfied} \\ 0 & \text{otherwise} \end{cases}$$

**** Four-point correlation numbers:

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \tilde{O}_{k_4} \rangle = \frac{\Sigma_{k_1 k_2 k_3 k_4}}{Z_p} \prod_{i=1}^4 \text{Leg}_L(k_i)$$

$$\Sigma_{k_1 \dots k_4} = (k_1 + 1)(p + k_1 + 3/2) - \sum_{i=2}^4 \sum_{s=-k_1:2}^{k_1} |p - 1/2 - k_i - s|$$

Applies when the number of conformal blocks in $\langle \Phi_{k_1} \dots \Phi_{k_4} \rangle$ is exactly k_1 . This holds for instance if

$$k_1 \leq k_2 \leq k_3 \leq k_4, \quad \text{and} \quad k_1 + k_4 \leq k_2 + k_3.$$

$$\sum_{s=-k_1:2}^{k_1} |p - \frac{1}{2} - k_i - s| = (k_1 + 1) \left(-2p + \frac{3}{2} + \sum_{i=1}^4 k_i \right) + \sum_{i=2}^4 \tilde{F}_p(k_1 + k_i),$$

$$\tilde{F}_p(k) = \frac{(p - k - 1)(p - k - 2)}{2} \Theta(k - p), \quad \Theta(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

*** Higher-point functions are (in principle) computable
[Belavin, Al.Zamolodchikov, unpublished]

- Generating function: $\{\tau\} = \{\tau_1, \tau_2, \dots, \tau_{p-1}\}$

$$W_{\mathcal{MG}}(\mu, \{\tau\}) = \left\langle \exp \left\{ - \sum_{i=1}^{p-1} \tau_i \tilde{O}_i \right\} \right\rangle_{\mathcal{MG}_{2/2p+1}}$$

The cosmological constant μ may be treated as $\mu = \tau_0$

$$S[\mathcal{MG}] = \dots + \underbrace{\mu \int e^{2b\varphi(x)} d^2x}_{\dots} + \dots$$

$$\tilde{O}_0 = \int V_0(x) \Phi_0(x) d^2x, \quad \Phi_0 = I$$

Dimensions:

$$\tau_k \sim \mu^{\frac{k+2}{2}}, \quad k = 0, 1, \dots, p-1$$

By the definition

$$\langle \tilde{O}_{k_1} \dots \tilde{O}_{k_n} \rangle = \frac{\partial^n W_{\mathcal{MG}}(\mu, \{\tau_i\})}{\partial \tau_{k_1} \dots \partial \tau_{k_n}} \Big|_{\{\tau_i\}=0}, \quad \{\tau_i\} = \{\tau_1, \dots, \tau_n\}$$

2. Matrix Models

Continuous (scaling) limit of the ensemble of planar graphs



Quantum Geometry

2.1. One-matrix Model

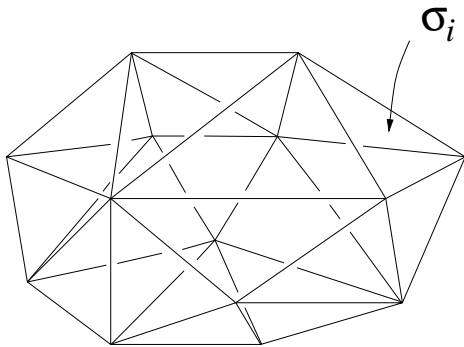
The planar graphs = Feynmann diagrams associated with the perturbative evaluation of the matrix integral

$$Z = \log \int dM e^{-N \text{tr} \left(\frac{1}{2} M^2 - \sum_{n=3} \frac{\alpha_n}{n!} M^n \right)}$$

M - Hermitian $N \times N$ matrix, N being the device for sorting out the topologies

$$Z = N^2 Z_0 + Z_1 + \dots + N^{2-2g} Z_g + \dots$$

Each term Z_g generates discretized surfaces, of the topology g , made of triangles and higher polygons, with the weights determined by α_i .



- We concentrate on $g = 0$ (sphere)

$$\Sigma \text{-space of the "potentials" } V(M) = \sum_{n=3} \frac{\alpha_n}{n!} M^n.$$

- The sum of the planar graphs exhibits *critical behavior*, in the vicinities of certain critical hyper-surfaces in Σ :

$$\dots \subset \Sigma_p \subset \dots \subset \Sigma_2 \subset \Sigma_1 \subset \Sigma$$



"p-criticality"

2.2. Solution of the One-matrix Model ($g = 0$)

- Result [*Brezin&Kazakov, 1990; Douglas&Shenker, 1990; Gross&Migdal, 1990*]: Near p -critical surface

$$u_* = u_*(t_0, \dots, t_{p-1}) = \frac{\partial^2 Z(t_0, \dots, t_{p-1})}{\partial t_{p-1}^2}$$

with u_* being certain solution of

$$Q(u) \equiv u^{p+1} - t_0 u^{p-1} - \dots - t_k u^{p-k-1} - \dots - t_{p-1} = 0$$

$\{t_0, t_1, \dots, t_{p-1}\}$ - deviations from Σ_p .

More convenient expression for Z :

$$Z = \frac{1}{2} \int_0^{u_*} Q^2(u) du .$$

- Interpretation [*Staudacher, 1990; Brezin&Douglas&Kazakov & Shenker, 1990; Gross&Migdal, 1990*]: Take

$$t_0 = \mu \quad \text{--"cosmological constant"}$$

Then

$$[u] = [\mu^{\frac{1}{2}}], \quad [t_k] = [\mu^{\frac{k+2}{2}}], \quad [Z] = [\mu^{\frac{2p+3}{2}}],$$

exactly the gravitational dimensions of $\mathcal{MG}_{2/2p+1}$,

$$t_k \sim \tau_k, \quad k = 0, 1, 2, \dots, p-1.$$

Convenient to separate $t_0 = \tilde{\mu}$ and $\{t_i\} = \{t_1, t_2, \dots, t_{p-1}\}$

Matrix Model correlation numbers:

$$\langle \mathcal{O}_{k_1} \dots \mathcal{O}_{k_n} \rangle_{MM} \equiv \frac{\partial^n W_{MM}(\mu, \{t_i\})}{\partial t_{k_1} \dots \partial t_{k_n}} \Big|_{\{t_i\}=0}, \quad \{t_i\} = \{t_1, \dots, t_n\}$$

with

$$W_{MM}(\mu, \{t_i\}) = \frac{Z(t_0 = \mu, t_1, \dots, t_n)}{Z(t_0 = \mu, 0, \dots, 0)}$$

With the (naive) identification

$$t_k \sim \tau_k$$

one would expect

$$\langle \mathcal{O}_{k_1} \dots \mathcal{O}_{k_n} \rangle_{MM} = \langle \tilde{\mathcal{O}}_{k_1} \dots \tilde{\mathcal{O}}_{k_n} \rangle_{\mathcal{MG}} \times [\text{Leg factors}]$$

This expectation fails.

Since

$$Q(u) = u^{p+1} - \mu u^{p-1} - \sum_{k=1}^{p-1} t_k u^{p-k-1}, \quad Z = \frac{1}{2} \int_0^{u_*} Q^2(u) du$$

we have $u_*(\mu, 0, \dots, 0) = \sqrt{\mu}$, and

$$\left. \frac{\partial Z}{\partial t_k} \right|_{\{t=0\}} = \int_0^{u_*} Q(u) \frac{\partial Q(u)}{\partial t_k} du \Big|_{\{t=0\}} = - \frac{2 \mu^{\frac{2p-k+1}{2}}}{(2p-k-1)(2p-k+1)}$$

$$\left. \frac{\partial^2 Z}{\partial t_k \partial t_{k'}} \right|_{\{t=0\}} = \int_0^{u_*} \frac{\partial Q(u)}{\partial t_k} \frac{\partial Q(u)}{\partial t_{k'}} du \Big|_{\{t=0\}} = \frac{\mu^{\frac{2p-k-k'-1}{2}}}{2p-k-k'-1}$$

etc

in sharp contrast with

$$\langle \tilde{\mathcal{O}}_k \rangle_{\mathcal{MG}} = 0, \quad k = 1, 2, \dots, p-1 \quad (\text{since } \langle \Phi_k \rangle_{CFT} = 0)$$

$$\langle \tilde{\mathcal{O}}_k \tilde{\mathcal{O}}_{k'} \rangle_{\mathcal{MG}} \sim \delta_{kk'}, \quad (\text{since } \langle \Phi_k \Phi_{k'} \rangle_{CFT} \sim \delta_{kk'})$$

etc

Resolution [Moore, Seiberg, Staudacher, 1991]: Resonances between the operators $\tilde{\mathcal{O}}_k$.

2.3. Resonance transformations

$$[t_k] = [\mu^{\frac{k+2}{2}}], \quad [\tau_k] = [\mu^{\frac{k+2}{2}}]$$

It is possible to have, e.g.

$$[t_k] = [\tau_{k_1}][\tau_{k_2}] \quad (k = k_1 + k_2 + 2 \geq 2)$$

($k = 0, 1, 2, \dots, p-1$). I.e.

$$t_k = \tau_k + \sum_{\substack{k_1, k_2=0 \\ k_1+k_2=k+2}}^{p-1} c_k^{k_1 k_2} \tau_{k_1} \tau_{k_2} + \text{higher order terms}$$

Thus

$$t_0 = \tau_0 = \mu,$$

$$t_1 = \tau_1, \quad ([t_1] = [\mu^{3/2}])$$

$$t_2 = \tau_2 + A_2 \mu^2, \quad ([t_2] = [\mu^2])$$

$$t_3 = \tau_3 + B_3 \mu \tau_1, \quad ([t_3] = [\mu][t_1])$$

$$t_4 = \tau_4 + A_4 \mu^3 + B_4 \mu \tau_2 + C_4 \tau_1^2$$

etc

generally

$$t_k = \tau_k + \underbrace{A_k \mu^{\frac{k+2}{2}}}_{\substack{\uparrow \\ k - \text{even}, \\ k \geq 2}} + \sum_{n=0}^{n \leq k/2} \underbrace{B_k^{k-2n} \mu^n \tau_{k-2n}}_{\substack{\uparrow \\ k \geq 3}} +$$

$$\frac{1}{2} \sum_{n=0} \sum_{k_1+k_2=k-2-2n} \underbrace{C_k^{k_1, k_2} \mu^n \tau_{k_1} \tau_{k_2}}_{\substack{\uparrow \\ k \geq 4}} + \dots$$

$$W_{MM}(\{t\}) \rightarrow \tilde{W}_{MM}(\{\tau\}) \equiv W_{MM}(\{t(\tau)\})$$

The right thing to expect is

$$\frac{\partial^N \tilde{W}_{MM}(\{\tau\})}{\partial \tau_{k_1} \dots \partial \tau_{k_N}} = \langle \tilde{O}_{k_1} \dots \tilde{O}_{k_N} \rangle_{\mathcal{MG}}$$

under special choice of the "Liouville coordinates" $\{\tau_1, \dots, \tau_n\}$.

$$\tau_k = 0, \quad k = 1, 2, \dots, p-1$$

↓

$$t_k = A_k \mu^{\frac{k+2}{2}} \quad - \text{ "Liouville background"}$$

Problem: Finding the "Liouville coordinates" $\{\tau\}$, such that

- One-point numbers:

$$\langle \tilde{O}_k \rangle_{MM} = \left. \frac{\partial \tilde{W}(\mu, \{\tau\})}{\partial \tau_k} \right|_{\{\tau\}=0} = 0 \quad \text{for } k = 1, 2, \dots, p-1$$

- Two-point numbers:

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle_{MM} = \left. \frac{\partial^2 \tilde{W}(\mu, \{\tau\})}{\partial \tau_k \partial \tau_{k'}} \right|_{\{\tau\}=0} \sim \delta_{kk'}$$

- Three-point numbers:

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \rangle_{MM} = \left. \frac{\partial^3 \tilde{W}(\mu, \{\tau\})}{\partial \tau_{k_1} \partial \tau_{k_2} \partial \tau_{k_3}} \right|_{\{\tau\}=0} = 0$$

obey the fusion rules.

- Multi-point numbers obey fusion rules, e.g.

For even $k_1 + \dots + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \dots \tilde{O}_{k_n} \rangle_{MM} = 0 \quad \text{if } k_n > k_1 + k_2 + \dots + k_{n-1}$$

For odd $k_1 + \dots + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \dots \tilde{O}_{k_n} \rangle_{MM} = 0 \quad \text{if } k_1 + k_2 + \dots + k_n < 2p-1$$

Building the Liouville coordinates order by order in $\{\tau\}$:

- The resonance transforms do not affect odd parity correlation functions.
- Starting from $n = 4$ there are not enough parameters to exterminate the "wrong" correlation numbers:

$$[\tau_k] = [\mu^{\frac{k+2}{2}}] \rightarrow [\tau_{k_1+k_2}] = [\tau_{k_1}][\tau_{k_2}][\mu^2]$$

2.4. Resonance terms in Z

$$[Z] = [\mu^{\frac{2p+3}{2}}]$$

Defined up to regular terms:

$$Z\{\tau\} \rightarrow Z(\{\tau\}) + \sum_n \sum_{k_1 \dots k_n} z^{k_1 \dots k_n} \underbrace{\tau_{k_1} \dots \tau_{k_n}}_{\uparrow}$$

$$[\tau_{k_1}][\tau_{k_2}] \dots [\tau_{k_n}] = [Z]$$

$$Z_{\text{reg}} = z_0 \mu^{\frac{2p+3}{2}} + z_k \mu^{\frac{2p-k+1}{2}} \tau_k + z_{k_1 k_2} \mu^{\frac{2p-k_1-k_2-1}{2}} \tau_{k_1} \tau_{k_2} + \dots$$

where only integer powers of μ are admitted.

The resonance terms affect negative parity correlation functions

$$\langle \tilde{O}_{k_1} \dots \tilde{O}_{k_n} \rangle_{MM} \quad \left(\sum_i k_i \text{ odd} \right)$$

with

$$k_1 + \dots + k_n \leq 2p + 3 - 2n$$

The fusion rules requires vanishing of the odd corr. numbers for

$$k_1 + \dots + k_n \leq 2p - 3$$

Again, starting from $n = 4$, there is not enough resonance terms to adjust.

3. Finding the Liouville coordinates

$$Q(u) = Q(u|\{\tau\}) = u^{p+1} - \mu u^{p-1} - \sum_{k=1}^{p-1} t_k(\{\tau\}) u^{p-k-1}$$

Expansion in $\{\tau\}$:

$$Q(u|\{\tau\}) = Q_0(u) + \sum_{k=1}^{p-1} \tau_k Q_k(u) + \frac{1}{2} \sum_{k_1, k_2=1}^{p-1} \tau_{k_1} \tau_{k_2} Q_{k_1 k_2}(u) + \dots$$

with

$$Q_0(u) = u^{p+1} - \sum_{l=1} A_l \mu^l u^{p+1-2l},$$

$$Q_k(u) = u^{p-k-1} - B_k^{(1)} \mu u^{p-k-3} - \dots - B_k^{(l)} \mu^l u^{p-k-2l-1} - \dots$$

$$Q_{k_1 k_2}(u) = -C_{k_1 k_2}^{(0)} u^{p-k_1-k_2-3} - C_{k_1 k_2}^{(1)} \mu u^{p-k_1-k_2-5} - \dots$$

etc

Parity (even or odd)

$$Q_0(-u) = (-)^{p+1} Q_0(u),$$

$$Q_k(-u) = (-)^{p+1-k} Q_k(u),$$

$$Q_{k_1 k_2}(-u) = (-)^{p+1-k_1-k_2} Q_{k_1 k_2}(u),$$

etc

$$Q_0(u) = Q(u)|_{\{\tau=0\}},$$

$$Q_k(u) = \left. \frac{\partial Q(u)}{\partial \tau_k} \right|_{\{\tau=0\}},$$

$$Q_{kk'}(u) = \left. \frac{\partial^2 Q(u)}{\partial \tau_k \partial \tau_{k'}} \right|_{\{\tau=0\}},$$

3.1. Partition function

$$Z(\mu, \{\tau\}) = \frac{1}{2} \int_0^{u_*} Q^2(u) du$$

where $u_* = u_*(\{\tau\})$ is some root of $Q(u)$

$$Q(u_*) = 0.$$

(All roots are real, and u_* is the maximal root)

3.2. One- and Two-point correlation numbers

$$\begin{aligned} \left. \frac{\partial Z}{\partial \tau_k} \right|_{\tau=0} &= \int_0^{u_0} Q(u) Q_k(u) du, \\ \left. \frac{\partial^2 Z}{\partial \tau_k \partial \tau_{k'}} \right|_{\tau=0} &= \int_0^{u_0} Q_k(u) Q_{k'}(u) du, \end{aligned}$$

where

$$Q_0(u_0) = 0.$$

For the corr. numbers with positive parity

$$\int_0^{u_0} \rightarrow \frac{1}{2} \int_{-u_*}^{u_*}$$

$$\begin{aligned} \int_{-u_0}^{u_0} Q(u) Q_k(u) du &= 0, \quad k = 1, \dots, p-1 \\ \int_{-u_0}^{u_0} [Q_k(u) Q_{k'}(u) + Q_0(u) Q_{kk'}(u)] du &\sim \delta_{kk'}, \end{aligned}$$

$$Q_0(u) = u_0^{p+1} q(u/u_0), \quad q_0(1) = 0,$$

$$Q_k(u) = u_0^{p-k-1} q(u/u_0), \quad \dots$$

$$\int_{-1}^1 q_0(x)q_k(x) dx = 0, \quad k = 1, \dots, p-1,$$

$$\int_{-1}^1 [q_k(x)q_{k'}(x) + q_0(x)q_{kk'}(x)] dx \sim \delta_{kk'}$$

Solution: the Legendre polynomials

$$q_k(x) = g_k P_{p-k-1}(x), \quad g_k = \frac{(p-k-1)!}{(2p-2k-3)!!}$$

$$q_0(x) = g_0 [P_{p+1}(x) - P_{p-1}(x)], \quad g_0 = \frac{(p+1)!}{(2p+1)!!}$$

↓

$$Z(\mu, \{\tau = 0\}) = \frac{1}{4} \int_{-u_0}^{u_0} Q_0^2(u) du = g_0^2 u_0^{2p+3} \frac{2p+1}{(2p-1)(2p+3)},$$

$$\langle \tilde{O}_k \rangle_{MM} = \frac{1}{2} \int_{-u_0}^{u_0} Q_0(u)Q_k(u) du = 0, \quad k = 1, 2, \dots, p-1$$

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle_{MM} = \frac{1}{2} \int_{-u_0}^{u_0} Q_k(u)Q_{k'}(u) du =$$

$$\frac{\delta_{kk'}}{Z_p} \frac{1}{2p-2k-1} [\text{Leg}(k)]^2$$

with $Z_p = [(2p-1)(2p+1)(2p+3)]^{-1}$,

$$\text{Leg}(k) = \frac{u_0^{-k-2}}{2p+1} \frac{g_k}{g_0}$$

3.3. Three-point correlation number

$$Z = \int_0^{u_*} Q^2(u) du$$

$$\begin{aligned} \left. \frac{\partial^3 Z}{\partial \tau_{k_1} \partial \tau_{k_2} \partial \tau_{k_3}} \right|_{\tau=0} &= - \frac{Q_{k_1}(u_0) Q_{k_2}(u_0) Q_{k_3}(u_0)}{Q'_0(u_0)} + \\ &\int_0^{u_0} [Q_{k_1}(u) Q_{k_2 k_3}(u) du + \text{two permutations}] du \end{aligned}$$

The integral cancels the "wrong" part of the first term iff

$$Q_{k_1 k_2}(u) = \frac{u_0^{-k_1-k_2-4}}{2p+1} \frac{g_{k_1} g_{k_2}}{g_0} \sum_{k=k_1+k_2+2:2}^{k \leq p-1} u_0^{k+2} \frac{2p-2k-1}{g_k} Q_k(u)$$

With this

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \rangle_{MM} = \frac{N_{k_1 k_2 k_3}}{Z_p} \prod_{i=1}^3 \text{Leg}(k_i).$$

3.4. Non-trivial test: four-point correlation numbers

$$\begin{aligned}
& \frac{\partial^4 Z}{\partial \tau_{k_1} \dots \partial \tau_{k_4}} = - \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q''}{(Q')^3} \Big|_{u=u_0} \\
& + \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q'_{k_4} + 3 \text{ permutations}}{(Q')^2} \Big|_{u=u_0} \\
& - \frac{Q_{k_1} Q_{k_2} Q_{k_3 k_4} + 5 \text{ permutations}}{Q'} \Big|_{u=u_0} \\
& + \int_0^{u_0} \left[Q_{k_1 k_2}(u) Q_{k_3 k_4}(u) + 2 \text{ permutations} \right] du \\
& + \underbrace{\int_0^{u_0} [Q_{k_1}(u) Q_{k_2 k_3 k_4}(u) + \text{permutations}] du}_{\substack{\uparrow \\ \text{"Counterterm"}}}
\end{aligned}$$

Evaluates to $\prod_{i=1}^4 \text{Leg}(k_i) \times$

$$\left[\sum_{i=1}^4 \frac{(p - k_i)(p - k_i - 1)}{2} - \frac{p(p+1)}{2} - F_p(k_{12|34}) - F_p(k_{13|24}) - F_p(k_{14|23}) \right]$$

with

$$F_p(k) = \frac{(p - k - 1)(p - k - 2)}{2} \Theta(p - k - 1)$$

and

$$k_{ij|nm} = \min(k_i + k_j, k_n + k_m)$$

- For $k_4 \leq k_1 + k_2 + k_3$ - reproduces exactly the four-point number of the Minimal Gravity $\mathcal{MG}_{2/2p+1}$

- At $k_4 > k_1 + k_2 + k_3$

$$\frac{(k_1 + k_2 + k_3 + 2 - k_4)(2p - 3 - k_1 - k_2 - k_3 - k_4)}{2} \prod_{i=1}^4 \text{Leg}(k_i)$$

Vanishes exactly at the "dangerous" configurations.

Conclusion

- The problem of finding the "Liouville coordinates" is rather rigid (leads to over-determined constraints on the coefficients of the resonance transformation and regular terms).
- At the level of 4-point numbers the solution exists.
- The resulting two, three, and four point numbers exactly reproduce the results of the Minimal Gravity.