Energy Correlations for a Random Matrix Model of Disordered Bosons

Martin R. Zirnbauer (Köln)

Landau Memorial Conference (June 22, 2008)

- Disordered bosons versus fermions
- The cone of elliptic elements
- A semi-realistic model, and how to analyze it
- Low energy effective field theory
- Solution of random matrix model

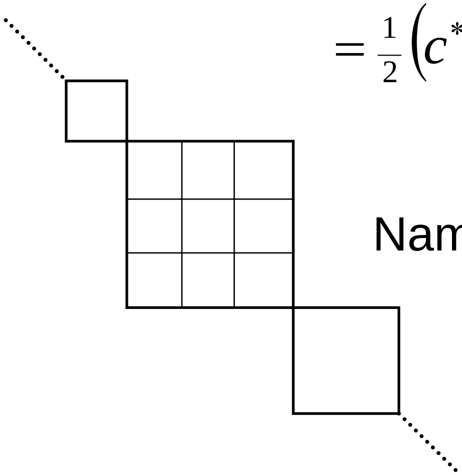
Symmetry classes of disordered fermions

Proof by Heinzner, Huckleberry, MRZ; Commun. Math. Phys. 257 (2005) 725
of the conjectured 10-fold way for the case of quadratic Hamiltonians:

$$H = \sum W_{ij}\, c_i^* c_j + \tfrac{1}{2} \sum (Z_{ij}\, c_i^* c_j^* + \text{h.c.})$$

$$= \tfrac{1}{2} \begin{pmatrix} c^* & c \end{pmatrix} \begin{pmatrix} W & Z \\ -\overline{Z} & -\overline{W} \end{pmatrix} \begin{pmatrix} c \\ c^* \end{pmatrix} \in \mathrm{i}\, \mathfrak{so}_{2N}(\mathbb{R})$$

$$(W = W^*,\ Z = -Z^{\mathrm{t}})$$



Nambu space $V \oplus V^*$ with symmetric bilinear form by CAR.

Unitary and anti-unitary symmetries: $G = U \cup TU$

(U compact)

Decompose into irreducible blocks
and transfer structure (CAR+T)

Every set of block data specifies a classical irreducible symmetric space,
and every classical irreducible symmetric space occurs in this way.

The ten large families of symmetric spaces

family	symmetric space	form of $H = \begin{pmatrix} W & Z \\ Z^* & -W^{t} \end{pmatrix}$
A	$\mathrm{U}(N)$	complex Hermitian
AI	$\mathrm{U}(N)/\mathrm{O}(N)$	real symmetric
AII	$\mathrm{U}(2N)/\mathrm{USp}(2N)$	quaternion self – adjoint
C	$\mathrm{USp}(2N)$	Z complex symmetric, $W = W^*$
CI	$\mathrm{USp}(2N)/\mathrm{U}(N)$	Z complex symmetric, $W = 0$
D	$\mathrm{SO}(2N)$	Z complex skew, $W = W^*$
DIII	$\mathrm{SO}(2N)/\mathrm{U}(N)$	Z complex skew, $W = 0$
AIII	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	Z complex $p\times q$, $W = 0$
BDI	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	Z real $p\times q$, $W = 0$
CII	$\mathrm{USp}(2p+2q)/\mathrm{USp}(2p)\times\mathrm{USp}(2q)$	Z quaternion $2p\times 2q$, $W = 0$

The 10-Way Table

For systems in the diffusive regime, random matrix models of a given symmetry class are known to be described by a corresponding supersymmetric nonlinear sigma model:

RME	A	AI	AII	C	CI	D	DIII	AIII	BDI	CII
noncomp.	AIII	BDI	CII	DIII	D	CI	C	A	AI	AII
susy NLsM										
compact	AIII	CII	BDI	CI	C	DIII	D	A	AII	AI

Efetov

For each class, the target space of the nonlinear sigma model is a Riemannian symmetric superspace

MRZ, J. Math. Phys. 37 (1996) 4986

Bosons are different

The universality question for low-frequency bosons

Disordered bosons with quadratic Hamiltonian :

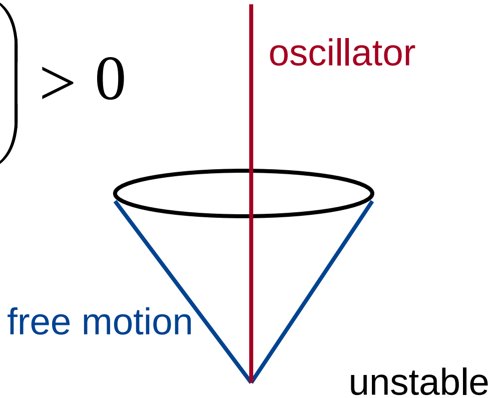
$$H = \sum W_{ij}\, a_i^* a_j + \tfrac{1}{2}\sum (Z_{ij}\, a_i^* a_j^* + \text{h.c.})$$

$$= \tfrac{1}{2}\begin{pmatrix} a^* & a \end{pmatrix} \begin{pmatrix} W & Z \\ -\overline{Z} & -\overline{W} \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix}, \qquad W = W^*,\ Z = +Z^{\mathrm{t}}.$$

$$\widetilde{H} := \begin{pmatrix} W & Z \\ -\overline{Z} & -\overline{W} \end{pmatrix} \in \mathrm{i}\,\mathfrak{sp}_{2N}(\mathbb{R})$$ (real symplectic Lie algebra)

Stability condition $\Sigma_3 \widetilde{H} = \begin{pmatrix} W & Z \\ \overline{Z} & \overline{W} \end{pmatrix} > 0$

defines cone of elliptic elements, E :



$$(a, a^*) \leftrightarrow (q, p)$$

$$\mathrm{i}\,\Sigma_3 \leftrightarrow J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Systems

- Vibrational modes of an amorphous solid
- Spin waves in a disordered magnet
- Density oscillations of a Bose-Einstein condensate subject to a random potential
- Electromagnetic modes in a disordered optical medium
- Excitations of a charge-density wave pinned by disorder

Note: because low-frequency Goldstone bosons have long wave length, they don't see weak disorder!

Chalker, Gurarie; Altland

The cone of elliptic elements

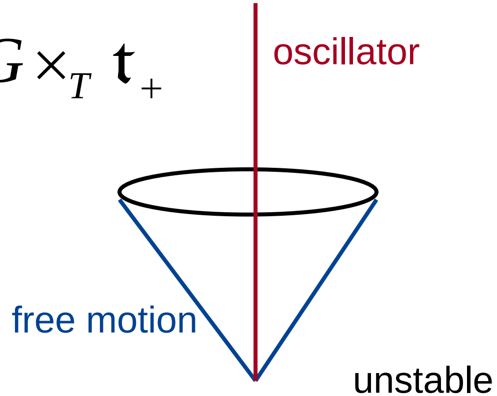
Notation: $G = \mathrm{Sp}_{2N}(\mathbb{R}), \ \mathfrak{g} = \mathfrak{sp}_{2N}(\mathbb{R})$

Maximal compact subgroup: $\mathrm{U}(N) \equiv K \subset G, \quad \mathfrak{k} = \mathrm{Lie}(K)$

Maximal torus: $T = \mathrm{U}(1)^N, \ \mathfrak{t} = \mathrm{Lie}(T).$

$E \subset \mathfrak{g}$ is an open, convex, $\mathrm{Ad}(G)-$invariant cone.

Geometric description: $E = (G/K) \times \mathfrak{k}_+ = G \times_T \mathfrak{t}_+$

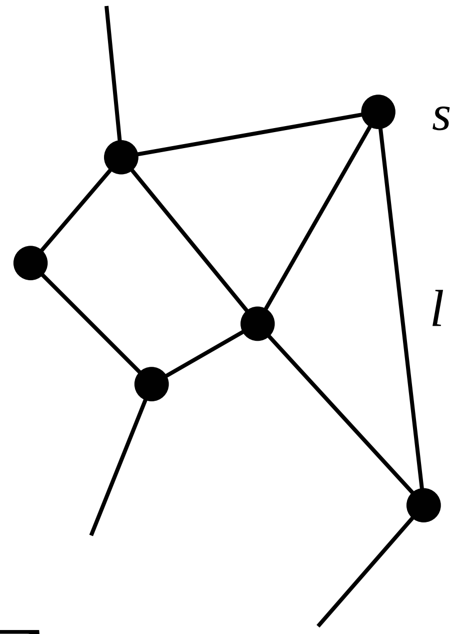


Desirable properties of RME:

1. probability density supported on E
2. locality of Hamiltonian (lattice, graph)
3. given 1.+2., postulate maximum entropy

The Model

Graph (or lattice) with sites s and links l.



Real symplectic vector spaces at sites :

$$W = \oplus W_s\,, \;\; \dim(W_s) = 2N_s$$

Auxiliary Euclidean vector spaces on links :

$$V = \oplus V_l\,, \;\; \dim(V_l) = M_l$$

Linear operators $L_{ls} : W_s \to V_l$

Generator of dynamics : $X = J\, L^{\mathrm{t}} L\,, \quad L = \sum_{s \in \partial l} L_{ls}$

Draw L_{ls} from Gaussian distribution with symmetry $\mathrm{O}(M_l) \times \mathrm{U}(N_s)$

Comparison with Wegner' s $n-$orbital model

How to analyze the model

Density of states $\leftarrow$ trace of resolvent :

$$\mathrm{Tr}\,(z - X)^{-1} = \mathrm{Tr}_W\,(z - J\,L^{\mathrm{t}} L)^{-1}$$

$$= \mathrm{Tr}_V\,(z - L\,J\,L^{\mathrm{t}})^{-1} + \frac{\dim W - \dim V}{z}.$$

Express trace of resolvent by determinants :

$$\mathrm{Tr}\,(z - L\,J\,L^{\mathrm{t}})^{-1} = \mathrm{Det}^{-1}\begin{pmatrix} z & -L \\ L^{\mathrm{t}} & J \end{pmatrix} \frac{d}{dz} \mathrm{Det}\begin{pmatrix} z & -L \\ L^{\mathrm{t}} & J \end{pmatrix}.$$

Write free – field representation of determinants.

Take ensemble average $\left\langle \mathrm{Tr}\,(z - X)^{-1} \right\rangle$.

Use superbosonization to construct supermatrix field theory.

Superbosonization

- P. Littelmann, H.-J. Sommers, and MRZ:
  Commun. Math. Phys. (published online: June 2008)
- J.E. Bunder, K.B. Efetov, V.E. Kravtsov, O.M. Yevtushenko, and MRZ: J. Stat. Phys. 129 (2007) 809

Lehmann, Saher, Sokolov, Sommers (95)

Hackenbroich, Weidenmüller (95)

Barruto, Brower, Svetitsky (01)

Efetov, Schwiete, Takahashi (04)

Guhr (06), Basile, Akemann (07)

Supersymmetry method (I)

$H = H^*$ linear operator on Hermitian vector space $\mathbb{C}^N$.

Gaussian integral over commuting variables $\varphi \in \mathbb{C}^N$:

$$\mathrm{Det}^{-1}(z-H) = \int \exp \mathrm{i}(\overline{\varphi}, \varphi z - H\varphi) , \qquad \mathrm{Im}\, z > 0 .$$

Gaussian $(\mathit{Berezin})$ integral over anti$-$commuting variables ψ :

$$\mathrm{Det}\,(w-H) = \int \exp -\mathrm{i}(\overline{\psi}, \psi\, w - H\psi) , \qquad w \in \mathbb{C} .$$

$$\left\langle \frac{\mathrm{Det}\,(w-H)}{\mathrm{Det}\,(z-H)} \right\rangle_{\mu} = \int \Omega\left(\varphi \otimes \overline{\varphi} + \psi \otimes \overline{\psi}\right) \exp \mathrm{i}\, z\, (\overline{\varphi}, \varphi) - \mathrm{i}\, w\, (\overline{\psi}, \psi) ,$$

characteristic function: $\Omega(K) = \int \exp\left(-\mathrm{i}\,\mathrm{Tr}\, HK\right) d\mu(H)$.

Supersymmetry method (II)

Characteristic function of the disorder distribution :

$\Omega(K) = \int \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,HK} d\mu(H)$, where ($\varphi$ commuting, ψ anticommuting variables) $\quad K_{ab} = \sum_{i=1}^{p} \varphi_{a,i}\,\tilde{\varphi}_{i,b} + \sum_{j=1}^{q} \psi_{a,j}\,\tilde{\psi}_{j,b}$.

Generating function for spectral correlation functions :

$\int_{\tilde{\varphi}=\varphi^*} D_{\varphi,\tilde{\varphi};\psi,\tilde{\psi}}\, f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) \equiv \int f$ where

$f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = \Omega(K)\, \exp\left(\mathrm{i}\sum_{i,a} \varphi_{a,i}\, z_i\, \tilde{\varphi}_{i,a} + \mathrm{i}\sum_{j,b} \psi_{b,j} w_j\, \tilde{\psi}_{j,b}\right)$.

Assume $d\mu(H)$ invariant by some group G acting by conjugation $H \mapsto gHg^{-1}$. Then $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}\,g^{-1};g\psi,\tilde{\psi}\,g^{-1})$.

Special case: commuting variables only

Let $p=1,\ q=0$ and consider GL_N −invariant holomorphic function $f:\mathbb{C}^N\times(\mathbb{C}^N)^*\to\mathbb{C},\ \ f(\varphi,\tilde{\varphi}) = f(g\varphi,\tilde{\varphi}\,g^{-1}),\ \ g\in\mathrm{GL}_N$.

Fact (from invariant theory): there exists a holomorphic function $F:\ \mathbb{C}\to\mathbb{C}$ such that $F(\tilde{\varphi}\cdot\varphi) = f(\varphi,\tilde{\varphi})$.

By push forward of the integral one has

$\int_{\mathbb{C}^N} f(\varphi,\varphi^*)\,d^{2N}\varphi = c_N\int_{\mathbb{R}_+} F(r)\,r^{N-1}dr$ (if the integral exists).

Generalization to $p>1$: see Fyodorov, Nucl. Phys. B 621 (2002) 643

Special case: Grassmann variables only

$p=0,\ q=1$:

Let $F:\mathbb{C}\to\mathbb{C}$ be a holomorphic function. For a vector ψ of anticommuting variables $\psi_1,\ldots,\psi_N$ consider $F(\tilde{\psi}\cdot\psi)$.

Berezin integral $\int F(\tilde{\psi}\cdot\psi)\, d\tilde{\psi}\, d\psi \ :=$

$$= \frac{\partial^2}{\partial\psi_1\partial\tilde{\psi}_1}\cdots\frac{\partial^2}{\partial\psi_N\partial\tilde{\psi}_N} F(\tilde{\psi}_1\psi_1+\ldots+\tilde{\psi}_N\psi_N)$$

$= F^{(N)}(0)$ (the N^{th} derivative at the origin)

$$= N! \oint_{\mathrm{U}(1)} F(z)\, z^{-N-1} dz / 2\pi\mathrm{i}.$$

q > 1: Kawamoto and Smit, Nucl. Phys. B **192** (1981) 100

Commuting and anticommuting variables

Recall $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) := f(g\varphi,\tilde{\varphi}\, g^{-1};g\psi,\tilde{\psi}\, g^{-1})$

for $g \in G$. Let $G = \mathrm{GL}_N$ or $G = \mathrm{O}_N$ or $G = \mathrm{Sp}_N$.

Superbosonization exploits this symmetry to make a step of reduction:

The integral over $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ of the G-invariant function f is converted to an integral over a Riemannian symmetric superspace. (The large number N of variables $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ then becomes a *parameter* of the integral.)

Example: $G = \mathrm{O}_N$

Lift $f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi})$ to $F(Q)$:

$$f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi}) = F\begin{pmatrix} \tilde{\varphi}\varphi & \tilde{\varphi}\tilde{\varphi}^{\mathrm{t}} & \tilde{\varphi}\psi & -\tilde{\varphi}\tilde{\psi}^{\mathrm{t}} \\ \varphi^{\mathrm{t}}\varphi & \varphi^{\mathrm{t}}\tilde{\varphi}^{\mathrm{t}} & \varphi^{\mathrm{t}}\psi & -\varphi^{\mathrm{t}}\tilde{\psi}^{\mathrm{t}} \\ \tilde{\psi}\varphi & \tilde{\psi}\tilde{\varphi}^{\mathrm{t}} & \tilde{\psi}\psi & -\tilde{\psi}\tilde{\psi}^{\mathrm{t}} \\ \psi^{\mathrm{t}}\varphi & \psi^{\mathrm{t}}\tilde{\varphi}^{\mathrm{t}} & \psi^{\mathrm{t}}\psi & -\psi^{\mathrm{t}}\tilde{\psi}^{\mathrm{t}} \end{pmatrix}$$

Theorem (LSZ). If $N \geq 2p$ and f Schwartz function along $\tilde{\varphi} = \varphi^*$, then

$$\int_{\tilde{\varphi}=\varphi^*} f = \int_M DQ \; \mathrm{SDet}^{N/2}(Q) \; F(Q)$$

with integration domain $M \cong (\mathrm{GL}_{2p}(\mathbb{R}) / \mathrm{O}_{2p}) \times (\mathrm{U}_{2q} / \mathrm{USp}_{2q})$

and $\mathfrak{gl}$ – invariant Berezin integration form DQ.

Remark : the supermanifold of integration is a

Riemannian symmetric superspace.

Supermatrix field theory

$p = q = 1, \quad n = M_l$

Matrix – valued field $l \mapsto Q_l$ $(4 \times 4$ supermatrices)

Target space $= \mathrm{GL}_{2|2} / \mathrm{OSp}_{2|2}$

over base $(\mathrm{GL}_2(\mathbb{R}) / \mathrm{O}_2) \times (\mathrm{U}_2 / \mathrm{USp}_2)$.

Partition function or (with source fields included) generating function :

$$\int \prod_l DQ_l \; \mathrm{e}^{-z\, \mathrm{STr}\, Q_l} \, \mathrm{SDet}^{M_l/2}(Q_l) \prod_s \mathrm{SDet}^{-N_s} (\Sigma_{l:\partial l \in s} \mathrm{var}(L_{ls}) Q_l - \mathrm{i}\Sigma_2) \; .$$

Low - energy effective theory is $\mathrm{GL}_{2|2} / \mathrm{OSp}_{2|2}$ nonlinear sigma model

with $\mathrm{GL}_{1|1}$ – gauge invariant coupling to 'matter' field.

(Analogy with gauge theory coupled to Higgs field)

Renormalization group?

Zero-dimensional limit $\longrightarrow$ one-matrix model

Zero-dimensional limit:

Solution of random matrix model

Random Matrix Model

Single site; maximal symmetry

$$d\mu(X) = e^{-\mathrm{Tr}(J^{-1}X)}\, \mathrm{Det}^{(\alpha-1)/2}(J^{-1}X)\; dX$$

$$X = J\, L^{\mathrm{t}} L, \;\; \alpha = \dim V - \dim W$$

Use Duistermaat-Heckman theorem to make reduction to joint probability distribution for eigenvalues $\omega_1, \ldots, \omega_N$:

$$d\mu_{N,\alpha}(\omega) = \prod_{i<j} (\omega_i - \omega_j)(\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \omega_k^{\alpha}\, e^{-\omega_k}\; d\omega_k$$

Exact solution using bi-orthogonal polynomials:

Lück, Sommers, MRZ; J. Math. Phys. 47 (2006) 103304

In the large-N limit, correlation functions exhibit GUE (sine kernel) universality at high frequencies ω and novel behavior for small ω .

High frequencies

Bulk scaling:

Scaling variable: $x = \omega / N$

Large-N limit of density of states:

$$\rho_\infty(x) = \tfrac{1}{\pi}\,\mathrm{Im}\, g(x + \mathrm{i}0); \qquad z = \frac{-2}{g(g+1)(g+2)}$$

can also be obtained from variational principle

('log gas')

Small-x behavior: $\rho_\infty(x) \propto x^{-1/3}$

Correlation functions for $N \to \infty$ in the bulk
obey GUE universality

Low frequencies

Edge scaling:

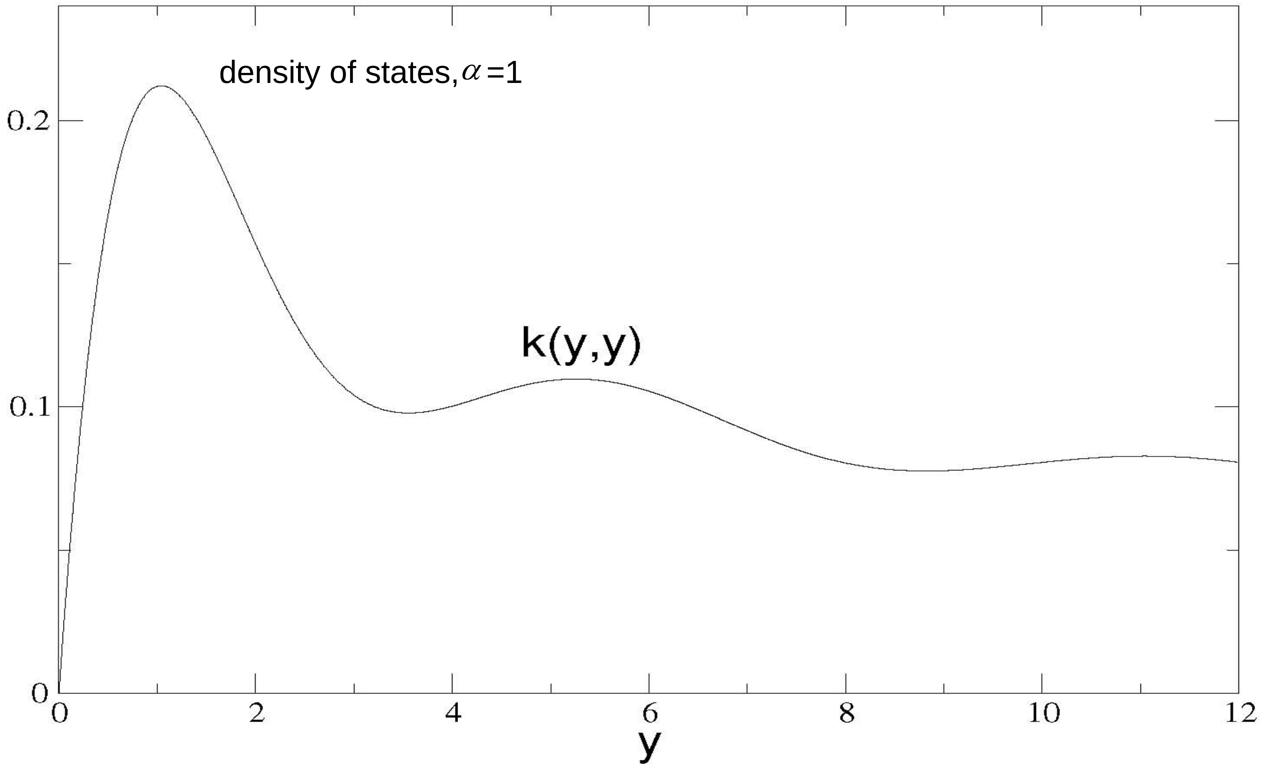
The good scaling variable now is $y = \sqrt{N}\omega$

Correln fcts are of determinantal type with kernel ($N \to \infty$)

$$k(y_1, y_2) = \frac{1}{2\pi^2} \int_{i\mathbb{R}+0} du \int_{U_1} dv \; e^{-y_1/u + y_2/v} v^{\alpha-1} u^{-\alpha} \frac{e^{u^2-v^2} - 1}{u^2 - v^2}$$

Different (but equivalent) expression for $k(y_1, y_2)$
was found by Borodin (1998)

Orthogonal polynomials were studied by Konhauser (1967)


density of states, α =1
k(y,y)
0.2
0.1
0
0
2
4
6
8
10
12
y

Discussion

- ‘Bosons are different’
- A graph-type model and its treatment by superbosonization
- Zero-dimensional limit:
  complete solution of invariant random matrix ensemble
  (novel type of correlation functions found)

Ongoing and future research:

- RG treatment of low-energy effective theory
- Develop analogous theory for time-reversal invariant systems
- Need further generalization to $H = H_0 + H_1$
  $H_0 =$ deterministic, $H_1 =$ random.